

HAWKING'S LOCAL RIGIDITY THEOREM WITHOUT ANALYTICITY

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ABSTRACT. We prove the existence of a Hawking Killing vector-field in a full neighborhood of a local, regular, bifurcate, non-expanding horizon embedded in a smooth vacuum Einstein manifold. The result extends a previous result of Friedrich, Rácz and Wald, see [7, Proposition B.1], which was limited to the domain of dependence of the bifurcate horizon. So far, the existence of a Killing vector-field in a full neighborhood has been proved only under the restrictive assumption of analyticity of the space-time. We also prove that, if the space-time possesses an additional Killing vectorfield \mathbf{T} , tangent to the horizon and not vanishing identically on the bifurcation sphere, then there must exist a local rotational Killing field commuting with \mathbf{T} . Thus the space-time must be locally axially symmetric. The existence of a Hawking vector-field \mathbf{K} , and the above mentioned axial symmetry, plays a fundamental role in the classification theory of stationary black holes. In [2] we use the results of this paper to prove a perturbative version of the uniqueness of smooth, stationary black holes in vacuum.

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1. INTRODUCTION

Let (\mathbf{M}, \mathbf{g}) to be a smooth¹ vacuum Einstein space-time. Let S be an embedded spacelike 2-sphere in \mathbf{M} and let $\mathcal{N}, \underline{\mathcal{N}}$ be the null boundaries of the causal set of S , i.e. the union of the causal future and past of S . We fix \mathcal{O} to be a small neighborhood of S such that both $\mathcal{N}, \underline{\mathcal{N}}$ are regular, achronal, null hypersurfaces in \mathcal{O} spanned by null geodesic generators orthogonal to S . We say that the triplet $(S, \mathcal{N}, \underline{\mathcal{N}})$ forms a

¹ \mathbf{M} is assumed to be a connected, oriented, C^∞ 4-dimensional manifold without boundary.

local, regular, bifurcate, non-expanding horizon in \emptyset if both $\mathcal{N}, \underline{\mathcal{N}}$ are non-expanding null hypersurfaces (see definition 2.1) in \emptyset . Our main result is the following:

Theorem 1.1. *Given a local, regular, bifurcate, non-expanding horizon $(S, \mathcal{N}, \underline{\mathcal{N}})$ in a smooth, vacuum Einstein space-time (\mathbf{O}, \mathbf{g}) , there exists an open neighborhood $\mathbf{O}' \subseteq \mathbf{O}$ of S and a non-trivial Killing vector-field \mathbf{K} in \mathbf{O}' , which is tangent to the null generators of \mathcal{N} and $\underline{\mathcal{N}}$. In other words, every local, regular, bifurcate, non-expanding horizon is a Killing bifurcate horizon.*

It is already known, see [7], that such a Killing vector-field exists in a small neighborhood of S intersected with the domain of dependence of $\mathcal{N} \cup \underline{\mathcal{N}}$. The extension of \mathbf{K} to a full neighborhood of S has been known to hold only under the restrictive additional assumption of analyticity of the space-time (see [8], [12], [7]). The novelty of our theorem is the existence of Hawking's Killing vector-field \mathbf{K} in a full neighborhood of the 2-sphere S , without making any analyticity assumption. It is precisely this information, i.e. the existence of \mathbf{K} in the complement of the domain of dependence of $\mathcal{N} \cup \underline{\mathcal{N}}$, that is needed in the application of Hawking's rigidity theorem to the classification theory of stationary, regular black holes. The assumption that the non-expanding horizon in Theorem 1.1 is *bifurcate* is essential for the proof; this assumption is consistent with the application mentioned above.

We also prove the following:

Theorem 1.2. *Assume that $(S, \mathcal{N}, \underline{\mathcal{N}})$ is a local, regular, bifurcate, horizon in a vacuum Einstein space-time (\mathbf{O}, \mathbf{g}) which possesses a Killing vectorfield \mathbf{T} tangent to $\mathcal{N} \cup \underline{\mathcal{N}}$ and non-vanishing on S . Then, there exists an open neighborhood $\mathbf{O}' \subseteq \mathbf{O}$ of S and a non-trivial rotational Killing vector-field \mathbf{Z} in \mathbf{O}' which commutes with \mathbf{T} .*

Once more, a related version of result was known only in the special case when the space-time is analytic. In fact S. Hawking's famous rigidity theorem, see [8], asserts that, under some global causality, asymptotic flatness and connectivity assumptions, a stationary, non-degenerate analytic spacetime must be axially symmetric. Observe that, though we have not assumed specifically that the horizon is non-expanding, this is in fact a well known consequence of the fact that the Killing field \mathbf{T} is tangent to it. Thus, in view of Theorem 1.1, there exists a Hawking vectorfield \mathbf{K} , in a full neighborhood of S . We show that there exist constants λ_0 and $t_0 > 0$ such that

$$\mathbf{Z} = \mathbf{T} + \lambda_0 \mathbf{K} \tag{1.1}$$

is a rotation with period t_0 . The main constants λ_0 and t_0 can be determined on the bifurcation sphere S . We remark that, though Hawking's rigidity theorem does not require, explicitly, a regular, bifurcate horizon, our assumption is related to that of the non-degeneracy of the event horizon, see [17].

As known the existence of the Hawking vector-field plays a fundamental role in the classification theory of stationary black holes (see [8] or [5] and references therein for a more complete treatment of the problem). The results of this paper are used in [2] to

prove a perturbative version, without analyticity, of the uniqueness of smooth, stationary black holes in vacuum. More precisely we show that a regular, smooth, asymptotically flat solution of the vacuum Einstein equations which is a perturbation of a Kerr solution $\mathcal{K}(a, m)$ with $0 \leq a < m$ is in fact a Kerr solution. The perturbation condition is expressed geometrically by assuming that the Mars-Simon tensor SS of the stationary space-time (see [15] and [10]) is sufficiently small. The proof uses Theorem 1.1 as a first step; one first defines a Hawking vector-field \mathbf{K} in a neighborhood of S and then extends it to the entire space-time by using the level sets of a canonically defined function y . One can show that these level sets are conditionally pseudo-convex, as in [10], as long as the Mars-Simon tensor SS is sufficiently small. Once \mathbf{K} is extended to the entire space-time one can show, using the result of Theorem 1.2, that the space-time is not only stationary but also axisymmetric. The proof then follows by appealing to the methods of the well known results of Carter [3] and Robinson [16], see also the more complete account [5].

1.1. Main Ideas. We recall that a Killing vector-field \mathbf{K} in a vacuum Einstein space-time must verify the covariant wave equation

$$\square_{\mathbf{g}} \mathbf{K} = 0. \quad (1.2)$$

The main idea in [7] was to construct \mathbf{K} as a solution to (1.2) with appropriate, characteristic, boundary conditions on $\mathcal{N} \cup \underline{\mathcal{N}}$. As known, the characteristic initial value problem is well posed in the domain of dependence of $\mathcal{N} \cup \underline{\mathcal{N}}$ but ill posed in its complement. To avoid this fundamental difficulty we rely instead on a completely different strategy². The main idea, which allows us to avoid using (1.2) or some other system of PDE's in the ill posed region, is to first construct \mathbf{K} in the domain of dependence of $\mathcal{N} \cup \underline{\mathcal{N}}$ as a solution to (1.2), extend \mathbf{K} by Lie dragging along the null geodesics transversal to \mathcal{N} , consider its associated flow Ψ_t , and show that, for small $|t|$, the pull back metric $\Psi_t^* \mathbf{g}$ must coincide with \mathbf{g} , in view of the fact they are both solutions of the Einstein vacuum equations and coincide on $\mathcal{N} \cup \underline{\mathcal{N}}$. To implement this idea we need to prove a uniqueness result for two Einstein vacuum metrics \mathbf{g}, \mathbf{g}' which coincide on $\mathcal{N} \cup \underline{\mathcal{N}}$. Such a uniqueness result was proved by one of the authors in [1], based on the uniqueness results for systems of covariant wave equations proved by the other two authors in [10] and [11]. The starting point of the proof are the schematic identities,

$$\square_{\mathbf{g}} \mathbf{R} = \mathbf{R} * \mathbf{R}, \quad \square_{\mathbf{g}'} \mathbf{R}' = \mathbf{R}' * \mathbf{R}'$$

with $\mathbf{R} * \mathbf{R}, \mathbf{R}' * \mathbf{R}'$ quadratic expressions in the curvatures \mathbf{R}, \mathbf{R}' of the Einstein vacuum metrics \mathbf{g}, \mathbf{g}' . Subtracting the two equations we derive,

$$\square_{\mathbf{g}}(\mathbf{R} - \mathbf{R}') + (\square_{\mathbf{g}} - \square_{\mathbf{g}'})\mathbf{R}' = (\mathbf{R} - \mathbf{R}') * (\mathbf{R} + \mathbf{R}').$$

² Such a strategy was discussed in [7, Remark B.1.], as an alternative to the use of the wave equation (1.2), in the domain of dependence of $\mathcal{N} \cup \underline{\mathcal{N}}$. We would like to thank R. Wald for drawing our attention to it.

We would like to rely on the uniqueness properties of covariant wave equations, as in [10], [11], but this is not possible due to the presence of the term $(\square_{\mathbf{g}} - \square_{\mathbf{g}'})\mathbf{R}'$ which forces us to consider equations for $\mathbf{g} - \mathbf{g}'$ expressed relative to an appropriate choice of a gauge condition. An obvious such gauge choice would be the wave gauge $\square_{\mathbf{g}}x^\alpha = 0$ which would lead to a system of wave equations for the components of the two metrics \mathbf{g}, \mathbf{g}' in the given coordinate system. Unfortunately such coordinate system would have to be constructed starting with data on $\mathcal{N} \cup \underline{\mathcal{N}}$ which requires one to solve the same ill posed problem. We rely instead on a pair of geometrically constructed frames v, v' (using parallel transport with respect to \mathbf{g} and \mathbf{g}') and derive ODE's for their difference $dv = v' - v$, as well as the difference $d\Gamma = \Gamma' - \Gamma$ between their connection coefficients, with source terms in $dR = \mathbf{R}' - \mathbf{R}$. In this way we derive a system of wave equations in dR coupled with ODE's in $dv, d\Gamma$ and their partial derivatives $\partial dv, \partial d\Gamma$ with respect to our fixed coordinate system. Since ODE's are clearly well posed it is natural to expect that the uniqueness results for covariant wave equations derived in [10], [11] can be extended to such coupled system and thus deduce that $dv = d\Gamma = dR = 0$ in a full neighborhood of S . The precise result is stated and proved in Lemma 4.4.

In section 2 we construct a canonical null frame which will be used throughout the paper. We use the non-expanding condition to derive the main null structure equations along \mathcal{N} and $\underline{\mathcal{N}}$. In section 3 we give a self contained proof of Proposition B.1. in [7] concerning the existence of a Hawking vector-field in the domain of dependence of $\mathcal{N} \cup \underline{\mathcal{N}}$. In section 4, we show how to extend \mathbf{K} to a full neighborhood of S . We also show that the extension must be locally time-like in the complement of the domain of dependence of $\mathcal{N} \cup \underline{\mathcal{N}}$, see Proposition 4.5. In section 5 we prove Theorem 1.2. We first show that if \mathbf{T} is another smooth Killing vector-field, tangent to $\mathcal{N} \cup \underline{\mathcal{N}}$, then it must commute with \mathbf{K} in a full neighborhood of S . We then construct a rotational Killing vector-field \mathbf{Z} as a linear combination of \mathbf{T} and \mathbf{K} . We also show that if σ_μ is the Ernst potential associated with \mathbf{T} then $\mathbf{K}^\mu = \mathbf{Z}^\mu \sigma_\mu = 0$. These additional results, in the presence of the (stationary) Killing vector-field \mathbf{T} , are important in the application in [2].

Acknowledgements: We would like to thank P. Chrusciel, M. Dafermos and R. Wald for helpful discussions and suggestions.

2. PRELIMINARIES

We restrict our attention to an open neighborhood \mathcal{O} of S in which $\mathcal{N}, \underline{\mathcal{N}}$ are regular, achronal, null hypersurfaces, spanned by null geodesic generators orthogonal to S . During the proof of our main theorem and their consequences we will keep restricting our attention to smaller and smaller neighborhoods of S ; for simplicity of notation we keep denoting such neighborhoods of S by \mathcal{O} .

We define two optical functions u, \underline{u} in a neighborhood of S as follows. We first fix a smooth future-directed null pair (L, \underline{L}) along S , satisfying

$$\mathbf{g}(L, L) = \mathbf{g}(\underline{L}, \underline{L}) = 0, \quad \mathbf{g}(L, \underline{L}) = -1, \quad (2.1)$$

such that L is tangent to \mathcal{N} and \underline{L} is tangent to $\underline{\mathcal{N}}$. In a small neighborhood of S , we extend L (resp. \underline{L}) along the null geodesic generators of \mathcal{N} (resp. $\underline{\mathcal{N}}$) by parallel transport, i.e. $\mathbf{D}_L L = 0$ (resp. $\mathbf{D}_{\underline{L}} \underline{L} = 0$). We define the function \underline{u} (resp. u) along \mathcal{N} (resp. $\underline{\mathcal{N}}$) by setting $u = \underline{u} = 0$ on S and solving $L(\underline{u}) = 1$ (resp. $\underline{L}(u) = 1$). Let $S_{\underline{u}}$ (resp. \underline{S}_u) be the level surfaces of \underline{u} (resp. u) along \mathcal{N} (resp. $\underline{\mathcal{N}}$). We define \underline{L} at every point of \mathcal{N} (resp. L at every point of $\underline{\mathcal{N}}$) as the unique, future directed null vector-field orthogonal to the surface $S_{\underline{u}}$ (resp. \underline{S}_u) passing through that point and such that $\mathbf{g}(L, \underline{L}) = -1$. We now define the null hypersurface $\underline{\mathcal{N}}_{\underline{u}}$ to be the congruence of null geodesics initiating on $S_{\underline{u}} \subset \mathcal{N}$ in the direction of \underline{L} . Similarly we define \mathcal{N}_u to be the congruence of null geodesics initiating on $\underline{S}_u \subset \underline{\mathcal{N}}$ in the direction of L . Both congruences are well defined in a sufficiently small neighborhood of S in \mathcal{O} , which (according to our convention) we continue to call \mathcal{O} . The null hypersurfaces $\underline{\mathcal{N}}_{\underline{u}}$ (resp. \mathcal{N}_u) are the level sets of a function \underline{u} (resp. u) vanishing on $\underline{\mathcal{N}}$ (resp. \mathcal{N}). By construction

$$L = -\mathbf{g}^{\mu\nu} \partial_\mu u \partial_\nu, \quad \underline{L} = -\mathbf{g}^{\mu\nu} \partial_\mu \underline{u} \partial_\nu. \quad (2.2)$$

In particular, the functions u, \underline{u} are both null optical functions, i.e.

$$\mathbf{g}^{\mu\nu} \partial_\mu u \partial_\nu u = \mathbf{g}(L, L) = 0 \quad \text{and} \quad \mathbf{g}^{\mu\nu} \partial_\mu \underline{u} \partial_\nu \underline{u} = \mathbf{g}(\underline{L}, \underline{L}) = 0. \quad (2.3)$$

We define,

$$\Omega = \mathbf{g}^{\mu\nu} \partial_\mu u \partial_\nu \underline{u} = \mathbf{g}(L, \underline{L}).$$

By construction $\Omega = -1$ on $(\mathcal{N} \cup \underline{\mathcal{N}}) \cap \mathcal{O}$, but Ω is not necessarily equal to -1 in \mathcal{O} . Choosing \mathcal{O} small enough, we may assume however that $\Omega \in [-3/2, -1/2]$ in \mathcal{O} .

To summarize, we can find two smooth optical functions $u, \underline{u} : \mathcal{O} \rightarrow \mathbb{R}$ such that,

$$\mathcal{N} \cap \mathcal{O} = \{p \in \mathcal{O} : u(p) = 0\}, \quad \underline{\mathcal{N}} \cap \mathcal{O} = \{p \in \mathcal{O} : \underline{u}(p) = 0\}. \quad (2.4)$$

and,

$$\Omega \in [-3/2, -1/2] \quad \text{in } \mathcal{O}. \quad (2.5)$$

Moreover, by construction (with L, \underline{L} defined by (2.2)) we have,

$$L(\underline{u}) = 1 \text{ on } \mathcal{N}, \quad \underline{L}(u) = 1 \text{ on } \underline{\mathcal{N}}.$$

Using the null pair \underline{L}, L introduced in (2.1), (2.2) we fix an associated null frame $e_1, e_2, e_3 = \underline{L}, e_4 = L$ such that $\mathbf{g}(e_a, e_a) = 1$, $\mathbf{g}(e_1, e_2) = \mathbf{g}(e_4, e_a) = \mathbf{g}(e_3, e_a) = 0$, $a = 1, 2$. At every point p in \mathcal{O} , e_1, e_2 form an orthonormal frame along the 2-surface $S_{u, \underline{u}}$ passing through p . We denote by ∇ the induced covariant derivative operator on $S_{u, \underline{u}}$. Given a horizontal vector-field X , i.e. X tangent to the 2-surfaces $S_{u, \underline{u}}$ at every point in \mathcal{O} , we denote by $\nabla_3 X, \nabla_4 X$ the projections of \mathbf{D}_{e_3} and \mathbf{D}_{e_4} to $S_{u, \underline{u}}$. Recall the definition of the null second fundamental forms

$$\chi_{ab} = \mathbf{g}(\mathbf{D}_{e_a} L, e_b), \quad \underline{\chi}_{ab} = \mathbf{g}(\mathbf{D}_{e_a} \underline{L}, e_b)$$

and the torsion

$$\zeta_a = \mathbf{g}(\mathbf{D}_{e_a} L, \underline{L}).$$

Definition 2.1. We say that \mathcal{N} is non-expanding if $\text{tr } \chi = 0$ on \mathcal{N} . Similarly $\underline{\mathcal{N}}$ is non-expanding if $\text{tr } \underline{\chi} = 0$ on $\underline{\mathcal{N}}$. The bifurcate horizon $(S, \mathcal{N}, \underline{\mathcal{N}})$ is called non-expanding if both $\mathcal{N}, \underline{\mathcal{N}}$ are non-expanding.

The assumption that the surfaces \mathcal{N} and $\underline{\mathcal{N}}$ are non-expanding implies, according to the Raychadhuri equation,

$$\chi = 0 \text{ on } \mathcal{N} \cap \emptyset \quad \text{and} \quad \underline{\chi} = 0 \text{ on } \underline{\mathcal{N}} \cap \emptyset. \quad (2.6)$$

In addition, since the vectors e_1, e_2 are tangent to $(\mathcal{N} \cup \underline{\mathcal{N}}) \cap \emptyset$ and $\mathbf{g}(L, \underline{L}) = -1$ on $(\mathcal{N} \cup \underline{\mathcal{N}}) \cap \emptyset$, we have $\zeta_a = -\mathbf{g}(D_{e_a} \underline{L}, L)$ on $(\mathcal{N} \cup \underline{\mathcal{N}}) \cap \emptyset$. Finally, it is known that the following components of the curvature tensor \mathbf{R} vanish on \mathcal{N} and $\underline{\mathcal{N}}$,

$$\mathbf{R}_{4a4b} = \mathbf{R}_{434b} = 0 \text{ on } \mathcal{N} \quad \text{and} \quad \mathbf{R}_{3a3b} = \mathbf{R}_{343b} = 0 \text{ on } \underline{\mathcal{N}}, \quad a, b = 1, 2. \quad (2.7)$$

Let, see [4], [14], $\alpha_{ab} = \mathbf{R}_{4a4b}$, $\beta_a = \mathbf{R}_{a434}$, $\rho = \mathbf{R}_{3434}$, $\sigma = {}^*\mathbf{R}_{3434}$, $\underline{\beta}_a = \mathbf{R}_{a334}$ and $\underline{\alpha}_{ab} = \mathbf{R}_{a3b3}$ denote the null components of \mathbf{R} . Thus, in view of (2.7) the only non-vanishing null components of \mathbf{R} on S are ρ and σ . Since $[e_a, e_4](\underline{L}) = 0$ on $\mathcal{N} \cap \emptyset$, it follows that $\mathbf{g}([e_a, e_4], e_3) = 0$ on $\mathcal{N} \cap \emptyset$. Using $\mathbf{D}_L L = 0$, (2.6), and the definitions, we derive, on $\mathcal{N} \cap \emptyset$,

$$\begin{aligned} \mathbf{D}_{e_4} e_4 &= 0, \quad \mathbf{D}_{e_a} e_4 = -\zeta_a e_4, \quad \mathbf{D}_{e_4} e_3 = -\sum_{a=1}^2 \zeta_a e_a, \quad \mathbf{D}_{e_4} e_a = \nabla_{e_4} e_a - \zeta_a e_4, \\ \mathbf{D}_{e_a} e_3 &= \sum_{b=1}^2 \underline{\chi}_{ab} e_b + \zeta_a e_3, \quad \mathbf{D}_{e_a} e_b = \nabla_{e_a} e_b + \underline{\chi}_{ab} e_4. \end{aligned} \quad (2.8)$$

Lemma 2.2. The null structure equations along \mathcal{N} (see³ Proposition 3.1.3 in [14]) reduce to

$$\nabla_4 \zeta = 0, \quad \text{curl } \zeta = \sigma, \quad L(\text{tr } \underline{\chi}) + \text{div } \zeta - |\zeta|^2 = \rho. \quad (2.9)$$

Also, if X is an horizontal vector,

$$[\nabla_4, \nabla_a] X_b = 0.$$

As a consequence we also have,

$$\nabla_4(\text{div } \zeta) = 0. \quad (2.10)$$

Proof of Lemma 2.2. Indeed,

$$\mathbf{g}(\mathbf{D}_4 \mathbf{D}_a \underline{L}, e_4) - \mathbf{g}(\mathbf{D}_a \mathbf{D}_4 \underline{L}, e_4) = \mathbf{R}(e_a, e_4, e_3, e_4) = \beta_a$$

and, using (2.8), $\mathbf{g}(\mathbf{D}_a \mathbf{D}_4 \underline{L}, e_4) = \underline{L}_{4;4a} = 0$, $\mathbf{g}(\mathbf{D}_4 \mathbf{D}_a \underline{L}, e_4) = \underline{L}_{4;a4} = -\nabla_4 \zeta_a$. Hence, since β vanishes along \mathcal{N} , we deduce $\nabla_4 \zeta = 0$. Also,

$$\mathbf{g}(\mathbf{D}_4 \mathbf{D}_b \underline{L}, e_a) - \mathbf{g}(\mathbf{D}_b \mathbf{D}_4 \underline{L}, e_a) = \mathbf{R}(e_a, e_3, e_4, e_b) = \frac{1}{2} \gamma_{ab} \rho - \frac{1}{2} \epsilon_{ab} \sigma$$

³The discrepancy with the corresponding formula is due to the different normalization for \underline{L} , i.e. $\mathbf{g}(L, \underline{L}) = -1$ instead of $\mathbf{g}(L, \underline{L}) = -2$.

and, $\mathbf{g}(\mathbf{D}_4 \mathbf{D}_b \underline{L}, e_a) = \underline{L}_{a;b4} = \nabla_4 \underline{X}_{ab} - 2\zeta_a \zeta_b$, $g(\mathbf{D}_b \mathbf{D}_4 \underline{L}, e_a) = \underline{L}_{a;4b} = -\nabla_b \zeta_a - \zeta_a \zeta_b$. Hence,

$$\nabla_4 \underline{X}_{ab} - \zeta_a \zeta_b + \partial_b \zeta_a = \frac{1}{2} \rho \gamma_{ab} - \frac{1}{2} \sigma \epsilon_{ab}.$$

Taking the symmetric part we derive, $\nabla_4 \text{tr} \underline{X} - |\zeta|^2 + \text{div} \zeta = \rho$ while taking the antisymmetric part yields, $\text{curl} \zeta = \sigma$ as desired. To check the commutation formula we write,

$$\begin{aligned} \mathbf{D}_4 \mathbf{D}_a X_b &= e_4(\mathbf{D}_a X_b) - \mathbf{D}_{\mathbf{D}_4 e_a} X_b - \mathbf{D}_a X_{\mathbf{D}_4 e_b} \\ &= e_4(\nabla_b X_a) - \mathbf{D}_{\nabla_4 e_a} X_b + \zeta_a \mathbf{D}_4 X_b - \mathbf{D}_a X_{\nabla_4 e_a} + \zeta_b \mathbf{D}_a X_4 \\ &= \nabla_4 \nabla_a X_b + \zeta_a \nabla_4 X_b \\ \mathbf{D}_a \mathbf{D}_4 X_b &= e_a(\mathbf{D}_4 X_b) - \mathbf{D}_{\mathbf{D}_a e_4} X_b - \mathbf{D}_4 X_{\mathbf{D}_a e_b} \\ &= e_a(\mathbf{D}_4 X_b) - \mathbf{D}_{\nabla_a e_4} X_b + \zeta_a \mathbf{D}_4 X_b - \mathbf{D}_4 X_{\nabla_a e_b} \\ &= \nabla_a \nabla_4 X_b + \zeta_a \nabla_4 X_b \end{aligned}$$

Therefore,

$$[\mathbf{D}_4, \mathbf{D}_a] X_b = [\nabla_4, \nabla_a] X_b.$$

On the other hand, $[\mathbf{D}_4, \mathbf{D}_a] X_b = \mathbf{R}_{a4cb} X^c = 0$ in view of the vanishing of β and the Einstein equations. \square

We define the following four regions I^{++} , I^{--} , I^{+-} and I^{-+} :

$$\begin{aligned} I^{++} &= \{p \in \emptyset : u(p) \geq 0 \text{ and } \underline{u}(p) \geq 0\}, & I^{--} &= \{p \in \emptyset : u(p) \leq 0 \text{ and } \underline{u}(p) \leq 0\}, \\ I^{+-} &= \{p \in \emptyset : u(p) \geq 0 \text{ and } \underline{u}(p) \leq 0\}, & I^{-+} &= \{p \in \emptyset : u(p) \leq 0 \text{ and } \underline{u}(p) \geq 0\}. \end{aligned} \quad (2.11)$$

Clearly I^{++}, I^{--} coincide with the causal and future and past sets of S in \emptyset .

3. CONSTRUCTION OF THE HAWKING VECTOR-FIELD IN THE CAUSAL REGION

We construct first the Killing vector-field \mathbf{K} in the causal region $I^{++} \cup I^{--}$.

Proposition 3.1. *Under the assumptions of Theorem 1.1, there is a small neighborhood \emptyset of S , a smooth Killing vector-field \mathbf{K} in $\emptyset \cap (I^{++} \cup I^{--})$ such that*

$$\mathbf{K} = \underline{u}L - u\underline{L} \quad \text{on } (\mathcal{N} \cup \underline{\mathcal{N}}) \cap \emptyset. \quad (3.1)$$

Moreover, in the region $\emptyset \cap (I^{++} \cup I^{--})$ where \mathbf{K} is defined, $[\underline{L}, \mathbf{K}] = -\underline{L}$.

The rest of this section is concerned with the proof of Proposition 3.1. The first part of the proposition, which depends on the main assumption that the surfaces \mathcal{N} and $\underline{\mathcal{N}}$ are non-expanding, is well known, see [7, Proposition B.1.]. For the sake of completeness, we provide its proof below.

Following [7] we construct the smooth vector-field \mathbf{K} as the solution to the characteristic initial-value problem,

$$\square_{\mathbf{g}}\mathbf{K} = 0, \quad \mathbf{K} = \underline{u}L - u\underline{L} \quad \text{on } (\mathcal{N} \cup \underline{\mathcal{N}}) \cap \emptyset. \quad (3.2)$$

As well known, see [18], the characteristic initial value problem for wave equations of type (3.3) is well posed. Thus the vector-field \mathbf{K} is well-defined and smooth in the domain of dependence of $\mathcal{N} \cup \underline{\mathcal{N}}$ in \emptyset . Let $\pi_{\alpha\beta} = {}^{(\mathbf{K})}\pi_{\alpha\beta} = \mathbf{D}_\alpha \mathbf{K}_\beta + \mathbf{D}_\beta \mathbf{K}_\alpha$. We have to prove that $\pi = 0$ in a neighborhood of S intersected to $I^{++} \cup I^{--}$. It follows from (3.2), using the Bianchi identities and the Einstein vacuum equations, that π verifies the covariant wave equation,

$$\square_{\mathbf{g}}\pi_{\alpha\beta} = 2\mathbf{R}^\mu{}_{\alpha\beta}{}^\nu \pi_{\mu\nu}. \quad (3.3)$$

In view of the standard uniqueness result for characteristic initial value problems, see [18], the statement of the proposition reduces to showing that $\pi = 0$ on $(\mathcal{N} \cup \underline{\mathcal{N}}) \cap \emptyset$. By symmetry, it suffices to prove that $\pi = 0$ on $\mathcal{N} \cap \emptyset$. The proof relies on our main hypothesis, that the surfaces \mathcal{N} and $\underline{\mathcal{N}}$ are non-expanding.

Since $\mathbf{K} = \underline{u}L$ on $\mathcal{N} \cap \emptyset$ is tangent to the null generators of \mathcal{N} , it follows that

$$\mathbf{D}_4 \mathbf{K}_3 = -1, \quad \mathbf{D}_4 \mathbf{K}_4 = \mathbf{D}_a \mathbf{K}_4 = \mathbf{D}_4 \mathbf{K}_a = \mathbf{D}_a \mathbf{K}_b = 0, \quad a, b = 1, 2. \quad (3.4)$$

Thus, on $\mathcal{N} \cap \emptyset$

$$\pi_{44} = \pi_{a4} = \pi_{ab} = 0 \quad a, b = 1, 2. \quad (3.5)$$

To prove that the remaining components of π vanish we use the wave equation $\square_{\mathbf{g}}\mathbf{K} = 0$, which gives

$$\mathbf{D}_3 \mathbf{D}_4 \mathbf{K}_\mu + \mathbf{D}_4 \mathbf{D}_3 \mathbf{K}_\mu = \sum_{a=1}^2 \mathbf{D}_a \mathbf{D}_a \mathbf{K}_\mu \quad \text{on } \mathcal{N} \cap \emptyset.$$

Since $\mathbf{D}_3 \mathbf{D}_4 \mathbf{K}_\mu - \mathbf{D}_4 \mathbf{D}_3 \mathbf{K}_\mu = \mathbf{R}_{34\mu\nu} \mathbf{K}^\nu$ on $\mathcal{N} \cap \emptyset$ (using (2.7)), we derive

$$2\mathbf{D}_4 \mathbf{D}_3 \mathbf{K}_\mu = \sum_{a=1}^2 \mathbf{D}_a \mathbf{D}_a \mathbf{K}_\mu - \mathbf{R}_{34\mu\nu} \mathbf{K}^\nu, \quad \mu = 1, 2, 3, 4, \quad \text{on } \mathcal{N} \cap \emptyset. \quad (3.6)$$

We set first $\mu = 4$. It follows from (3.4) that $\mathbf{D}_4 \mathbf{D}_3 \mathbf{K}_4 = 0$. In addition, $\mathbf{D}_3 \mathbf{K}_4 = 1$ on S (the analogue of the first identity in (3.4) along the hypersurface $\underline{\mathcal{N}}$). Using (2.8) and (3.4), $\mathbf{D}_4 \mathbf{D}_3 \mathbf{K}_4 = L(\mathbf{D}_3 \mathbf{K}_4)$. Thus $\mathbf{D}_3 \mathbf{K}_4 = 1$ on \mathcal{N} , which implies

$$\pi_{34} = 0 \quad \text{on } \mathcal{N}. \quad (3.7)$$

We use now the equation (3.6) with $\mu = a \in \{1, 2\}$ to calculate $P_a := \pi_{a3}$ along \mathcal{N} . It follows from (3.4) and (2.7) that $\mathbf{D}_a \mathbf{D}_b \mathbf{K}_c = 0$, $a, b, c = 1, 2$, and $\mathbf{R}_{34a\nu} \mathbf{K}^\nu = 0$ on \mathcal{N} . A simple computation shows that $\mathbf{D}_a \mathbf{K}_3 = \underline{u}\zeta_a$, thus $P_a = \mathbf{D}_3 \mathbf{K}_a + \underline{u}\zeta_a$. Thus, using (2.8), $\mathbf{D}_3 \mathbf{K}_4 = 1$, and $\mathbf{D}_b \mathbf{K}_c = 0$ on \mathcal{N} , we derive

$$\begin{aligned} 0 &= \mathbf{K}_{b;34} = e_4(\mathbf{K}_{b;3}) - \mathbf{K}_{\mathbf{D}_{e_4}e_b;e_3} - \mathbf{K}_{e_b;\mathbf{D}_{e_4}e_3} = e_4(P_b - \underline{u}\zeta_b) - \mathbf{K}_{\nabla_4 e_b;e_3} + \zeta_b K_{e_4;e_3} \\ &= \nabla_4(P_b - \underline{u}\zeta_b) + \zeta_b = \nabla_4 P_b - \underline{u}\nabla_4 \zeta_b. \end{aligned}$$

Thus

$$\nabla_4 P_a = \underline{u} \nabla_4 \zeta_a \quad \text{on } \mathcal{N}.$$

On the other hand, along \mathcal{N} , ζ verifies the transport equation,

$$\nabla_4 \zeta_a = -\mathbf{R}_{a434} = 0.$$

Therefore, along \mathcal{N} ,

$$\nabla_4 P_a = 0.$$

Since $P_a = \pi_{a3} = 0$ on S it follows that

$$\pi_{a3} = 0 \quad \text{on } \mathcal{N}. \quad (3.8)$$

Similarly, denoting $Q = \pi_{33} = 2\mathbf{D}_3 \mathbf{K}_3$, we have, according to (3.6) with $\mu = 3$,

$$\mathbf{D}_4 \mathbf{D}_3 \mathbf{K}_3 = \frac{1}{2} \left(\sum_{a=1}^2 \mathbf{D}_a \mathbf{D}_a \mathbf{K}_3 - \rho \underline{u} \right), \quad \rho = \mathbf{R}_{3434}. \quad (3.9)$$

Now, since we already now that π_{3b} vanishes on \mathcal{N} ,

$$\mathbf{K}_{3;34} = e_4(\mathbf{K}_{3;3}) - \mathbf{K}_{\mathbf{D}_{e_4} e_3; e_3} - \mathbf{K}_{e_3; \mathbf{D}_{e_4} e_3} = \frac{1}{2} e_4(Q) + \sum_{b=1}^2 \zeta_b \pi_{3b} = \frac{1}{2} e_4(Q). \quad (3.10)$$

On the other hand, using (2.8), $\mathbf{K}_{3;4} = -1$, $\mathbf{K}_{a;b} = 0$, and $\mathbf{K}_{3;a} = \underline{u} \zeta_a$,

$$\begin{aligned} \mathbf{K}_{3;ab} &= e_b(\mathbf{K}_{3;a}) - \mathbf{K}_{e_3; \mathbf{D}_{e_b} e_a} - \mathbf{K}_{\mathbf{D}_{e_b} e_3; e_a} \\ &= \partial_b(\underline{u} \zeta_a) - \underline{\chi}_{ba} \mathbf{K}_{3;4} - \zeta_b \mathbf{K}_{3;a} \\ &= \partial_b(\underline{u} \zeta_a) + \underline{\chi}_{ba} - \underline{u} \zeta_a \zeta_b, \end{aligned}$$

thus

$$\sum_{a=1}^2 \mathbf{D}_a \mathbf{D}_a \mathbf{K}_3 = \underline{u}(\operatorname{div} \zeta - |\zeta|^2) + \operatorname{tr} \underline{\chi}. \quad (3.11)$$

Therefore, equation (3.9) takes the form

$$L(Q) = \operatorname{tr} \underline{\chi} + \underline{u}(\operatorname{div} \zeta - |\zeta|^2 - \rho). \quad (3.12)$$

On the other hand we have the following structure equation on \mathcal{N} ,

$$L(\operatorname{tr} \underline{\chi}) + \operatorname{div} \zeta - |\zeta|^2 - \rho = 0. \quad (3.13)$$

Thus, differentiating (3.12) with respect to L and applying (3.13) we derive,

$$\begin{aligned} L(L(Q)) &= L(\operatorname{tr} \underline{\chi}) + (\operatorname{div} \zeta - |\zeta|^2 - \rho) + \underline{u} L(\operatorname{div} \zeta - |\zeta|^2 - \rho) \\ &= -\operatorname{div} \zeta + |\zeta|^2 + \rho + (\operatorname{div} \zeta - |\zeta|^2 - \rho) + \underline{u} L(\operatorname{div} \zeta - |\zeta|^2 - \rho). \end{aligned}$$

Using null structure equations, it is not hard to check that

$$L(\operatorname{div} \zeta) = L(|\zeta|^2) = L(\rho) = 0 \quad \text{along } \mathcal{N}. \quad (3.14)$$

Indeed, the last identity follows from (2.7) and [14, Proposition 3.2.4]. The identity $L(|\zeta|^2) = 0$ follows from the transport equation $\nabla_4 \zeta_a = 0$. Therefore,

$$L(L(Q)) = 0 \quad \text{along } \mathcal{N}.$$

Since $L(Q) = 0$ on S (using again (3.12) restricted to S where both $\text{tr } \underline{\chi}$ and \underline{u} vanish), we infer that $L(Q) = 0$ along \mathcal{N} . Since $Q = 0$ on S we conclude that $Q = 0$ along \mathcal{N} as desired. Thus $\pi_{33} = 0$, as desired.

The second part of the proposition, $[\underline{L}, \mathbf{K}] = -\underline{L}$ in a neighborhood of S in $I^{++} \cup I^{--}$, follows from the identity,

$$\mathbf{D}_{\underline{L}} W = -\mathbf{D}_W \underline{L} \quad \text{where} \quad W = [\underline{L}, \mathbf{K}] + \underline{L} = -\mathcal{L}_{\mathbf{K}} \underline{L} + \underline{L},$$

and the vanishing of W on $\mathcal{N} \cap \emptyset$. To prove the identity we make use of the fact that $\mathcal{L}_{\mathbf{K}}$ commutes with covariant differentiation. In particular, if \mathbf{K} is Killing and X, Y arbitrary vector-fields then,

$$\mathcal{L}_{\mathbf{K}}(\mathbf{D}_X Y) = \mathbf{D}_X(\mathcal{L}_{\mathbf{K}} Y) + \mathbf{D}_{\mathcal{L}_{\mathbf{K}} X} Y. \quad (3.15)$$

Therefore,

$$\mathbf{D}_{\underline{L}} W = \mathbf{D}_{\underline{L}}(-\mathcal{L}_{\mathbf{K}} \underline{L} + \underline{L}) = -\mathbf{D}_{\underline{L}} \mathcal{L}_{\mathbf{K}} \underline{L} = \mathcal{L}_{\mathbf{K}}(\mathbf{D}_{\underline{L}} \underline{L}) + \mathbf{D}_{(\mathcal{L}_{\mathbf{K}} \underline{L})} \underline{L} = -\mathbf{D}_W \underline{L}.$$

as stated. It remains to prove that

$$W = [\underline{L}, \mathbf{K}] + \underline{L} = 0 \quad \text{on } \mathcal{N} \cap \emptyset. \quad (3.16)$$

Since $\mathbf{K} = \underline{u}L$ on $\mathcal{N} \cap \emptyset$, this is equivalent to

$$\mathbf{D}_3 \mathbf{K}_\mu - \underline{u} \mathbf{D}_4 \underline{L}_\mu + \underline{L}_\mu = 0 \quad \text{on } \mathcal{N} \cap \emptyset, \quad \mu = 1, 2, 3, 4. \quad (3.17)$$

We check (3.17) on the null frame $e_1, e_2, e_3 = \underline{L}, e_4 = L$ defined earlier. The identity (3.17) follows for $\mu = a = 1, 2$ since $\mathbf{D}_3 \mathbf{K}_a = -\mathbf{D}_a \mathbf{K}_3 = -\underline{u} \zeta_a$, $\mathbf{D}_4 \underline{L}_a = \mathbf{g}(e_a, \mathbf{D}_{e_4} e_3) = -\zeta_a$ (see (2.8)), and $\underline{L}_a = 0$. The identity also follows for $\mu = 3$ since $\mathbf{D}_3 \mathbf{K}_3 = \pi_{33}/2 = 0$ (in view of Proposition 3.1), $\mathbf{D}_4 \underline{L}_3 = \mathbf{g}(e_3, \mathbf{D}_{e_4} e_3) = 0$ (see (2.8)), and $\underline{L}_3 = 0$. Finally, for $\mu = 4$, $\mathbf{D}_3 \mathbf{K}_4 = -\mathbf{D}_4 \mathbf{K}_3 = 1$ (see (3.4)), $\mathbf{D}_4 \underline{L}_4 = \mathbf{g}(e_4, \mathbf{D}_{e_4} e_3) = 0$, and $\underline{L}_4 = -1$. This completes the proof of the proposition.

4. EXTENSION OF THE HAWKING VECTOR-FIELD TO A FULL NEIGHBORHOOD

In the previous section we have defined our Hawking vector-field \mathbf{K} in a neighborhood \emptyset of S intersected with $I^{++} \cup I^{--}$. To extend \mathbf{K} in the exterior region $I^{+-} \cup I^{-+}$ we cannot rely on solving equation (3.2); the characteristic initial value problem is ill posed in that region. We need to rely instead on a completely different strategy, sketched in the introduction. We extend \mathbf{K} by Lie dragging it relative to \underline{L} and show that, for small $|t|$, $\Psi_t^* \mathbf{g}$ must coincide with \mathbf{g} , where $\Psi_t = \Psi_{t, \mathbf{K}}$ is the flow generated by \mathbf{K} . We show that both metrics coincide on $\mathcal{N} \cup \underline{\mathcal{N}}$ and, since they both verify the vacuum Einstein equations, we prove that they must coincide in a full neighborhood of S .

To implement this strategy we first define the vector-field K' by setting $K' = \underline{u}L$ on $\mathcal{N} \cap \emptyset$ and solving the ordinary differential equation $[\underline{L}, K'] = -\underline{L}$. The vector-field K'

is well-defined and smooth in a small neighborhood of S (since $\underline{L} \neq 0$ on S) and coincides with \mathbf{K} in $I^{++} \cup I^{--}$ in \emptyset . Thus $\mathbf{K} := K'$ defines the desired extension. This proves the following.

Lemma 4.1. *There exists a smooth extension of the vector-field \mathbf{K} (defined in Proposition 3.1) to an open neighborhood \emptyset of S such that*

$$[\underline{L}, \mathbf{K}] = -\underline{L} \quad \text{in } \emptyset. \quad (4.1)$$

It remains to prove that \mathbf{K} is indeed our desired Killing vector-field. For $|t|$ sufficiently small, we define, in a small neighborhood of S , the map $\Psi_t = \Psi_{t, \mathbf{K}}$ obtained by flowing a parameter distance t along the integral curves of \mathbf{K} . Let

$$\mathbf{g}^t = \Psi_t^*(\mathbf{g}).$$

The Lorentz metrics \mathbf{g}^t are well-defined in a small neighborhood of S , for $|t|$ sufficiently small. To show that \mathbf{K} is Killing we need to show that in fact $\mathbf{g}^t = \mathbf{g}$. Since \mathbf{K} is tangent to $(\mathcal{N} \cup \underline{\mathcal{N}}) \cap \emptyset$ and is Killing in $I^{++} \cup I^{--}$, we infer that $\mathbf{g}^t = \mathbf{g}$ in a small neighborhood of S intersected with $I^{++} \cup I^{--}$. In view of the definition of \mathbf{K} (see (4.1)),

$$\frac{d}{dt} \Psi_t^* \underline{L} = \lim_{h \rightarrow 0} \frac{\Psi_{t-h}^* \underline{L} - \Psi_t^* \underline{L}}{-h} = -\Psi_t^* \left(\lim_{h \rightarrow 0} \frac{\Psi_{-h}^* \underline{L} - \Psi_0^* \underline{L}}{h} \right) = -\Psi_t^* (\mathcal{L}_{\mathbf{K}} \underline{L}) = -\Psi_t^* \underline{L}.$$

We infer that,

$$\Psi_t^* \underline{L} = e^{-t} \underline{L}.$$

Now, given arbitrary vector-fields X, Y , we have $\mathbf{D}_{X^t}^t Y^t = \Psi_t^*(\mathbf{D}_X Y)$ where \mathbf{D}^t denotes the covariant derivative induced by the metric $\mathbf{g}^t = \Psi_t^* g$ and $X^t = \Psi_t^* X$, $Y^t = \Psi_t^* Y$. In particular $0 = \mathbf{D}_{\underline{L}^t}^t \underline{L}^t = e^{-2t} \mathbf{D}_{\underline{L}}^t \underline{L}$. This proves the following.

Lemma 4.2. *Assume \mathbf{K} is a smooth vector-field verifying (4.1) and \mathbf{D}^t the covariant derivative induced by the metric $\mathbf{g}^t = \Psi_t^* g$. Then,*

$$\mathbf{D}_{\underline{L}}^t \underline{L} = 0 \quad \text{in a small neighborhood of } S.$$

To summarize we have a family of metrics \mathbf{g}^t which verify the Einstein vacuum equations $\mathbf{Ric}(\mathbf{g}^t) = 0$, $\mathbf{g}^t = \mathbf{g}$ in a small neighborhood of S intersected with $I^{++} \cup I^{--}$, and such that $\mathbf{D}_{\underline{L}}^t \underline{L} = 0$. Without loss of generality we may assume that both relations hold in \emptyset . Thus Theorem 1.1 is an immediate consequence of the following:

Proposition 4.3. *Assume \mathbf{g}' is a smooth Lorentz metric on \emptyset , such that (\emptyset, \mathbf{g}') is a smooth Einstein vacuum space-time. Assume that*

$$\mathbf{g}' = \mathbf{g} \quad \text{in } (I^{++} \cup I^{--}) \cap \emptyset \quad \text{and} \quad \mathbf{D}'_{\underline{L}} \underline{L} = 0 \quad \text{in } \emptyset,$$

where \mathbf{D}' denotes the covariant derivative induced by the metric \mathbf{g}' . Then $\mathbf{g}' = \mathbf{g}$ in a small neighborhood $\emptyset' \subset \emptyset$ of S .

As explained in the introduction, this proposition was first proved in [1]. We provide here a more direct, simpler proof, specialized to our situation and based on the uniqueness result in Lemma 4.4 below. That lemma is an extension of the uniqueness results proved in [10] to coupled systems of covariant wave equations and ODE's. The motivation for the proof below was given in the introduction.

Proof of Proposition 4.3. It suffices to prove the proposition in a neighborhood $\mathcal{O}(x_0)$ of a point x_0 in S in which we can introduce a fixed coordinate system x^α . Without loss of generality we may assume that

$$\mathbf{g}_{ij}(x_0) = \text{diag}(-1, 1, 1, 1), \quad \sup_{x \in \mathcal{O}(x_0)} \sum_{j=0}^6 |\partial^j \mathbf{g}(x)| \leq A, \quad (4.2)$$

with $|\partial^j \mathbf{g}|$ denoting the sum of the absolute values of all partial derivatives of order j for all components of \mathbf{g} in the given coordinate system. We may also assume, for the optical functions u, \underline{u} introduced in section 2,

$$\sup_{x \in \mathcal{O}(x_0)} (|\partial^j u(x)| + |\partial^j \underline{u}(x)|) \leq C_1 = C_1(A) \quad \text{for } j = 0, \dots, 4. \quad (4.3)$$

In the rest of the proof we will keep restricting to smaller and smaller neighborhoods of x_0 ; for simplicity of notation we keep denoting such neighborhoods by $\mathcal{O}(x_0)$.

Consider now our old null frame $\tilde{v}_{(1)} = e_1, \tilde{v}_{(2)} = e_2, \tilde{v}_{(3)} = L, \tilde{v}_{(4)} = \underline{L}$ on $\mathcal{N} \cap \mathcal{O}(x_0)$ and define the vector-fields $v_{(1)}, v_{(2)}, v_{(3)}, v_{(4)} = \underline{L}$ and $v'_{(1)}, v'_{(2)}, v'_{(3)}, v'_{(4)} = \underline{L}$ by parallel transport along \underline{L} :

$$\begin{aligned} \mathbf{D}_{\underline{L}} v_{(a)} &= 0 \text{ and } v_{(a)} = \tilde{v}_a \text{ on } \mathcal{N} \cap \mathcal{O}(x_0); \\ \mathbf{D}'_{\underline{L}} v'_{(a)} &= 0 \text{ and } v'_{(a)} = \tilde{v}_a \text{ on } \mathcal{N} \cap \mathcal{O}(x_0). \end{aligned}$$

The vector-fields $v_{(a)}$ and $v'_{(a)}$ are well-defined and smooth in $\mathcal{O}(x_0)$. Let $\mathbf{g}_{(a)(b)} = \mathbf{g}(v_{(a)}, v_{(b)})$, $\mathbf{g}'_{(a)(b)} = \mathbf{g}(v'_{(a)}, v'_{(b)})$. The identities $\mathbf{D}_{\underline{L}} v_{(a)} = \mathbf{D}'_{\underline{L}} v'_{(a)} = 0$ show that $\underline{L}(\mathbf{g}_{(a)(b)}) = \underline{L}(\mathbf{g}'_{(a)(b)}) = 0$. Since $\mathbf{g}_{(a)(b)} = \mathbf{g}'_{(a)(b)}$ along \mathcal{N} it follows that

$$\mathbf{g}_{(a)(b)} = \mathbf{g}'_{(a)(b)} := h_{(a)(b)} \text{ and } \underline{L}(h_{(a)(b)}) = 0 \text{ in } \mathcal{O}(x_0). \quad (4.4)$$

For $a, b, c = 1, \dots, 4$ let

$$\begin{aligned} \Gamma_{(a)(b)(c)} &= \mathbf{g}(v_{(a)}, \mathbf{D}_{v_{(c)}} v_{(b)}), \quad \Gamma'_{(a)(b)(c)} = \mathbf{g}'(v'_{(a)}, \mathbf{D}'_{v'_{(c)}} v'_{(b)}), \\ (d\Gamma)_{(a)(b)(c)} &= \Gamma'_{(a)(b)(c)} - \Gamma_{(a)(b)(c)}. \end{aligned}$$

For $a, b, c, d = 1, \dots, 4$ let

$$\begin{aligned} \mathbf{R}_{(a)(b)(c)(d)} &= \mathbf{R}(v_{(a)}, v_{(b)}, v_{(c)}, v_{(d)}), \quad \mathbf{R}'_{(a)(b)(c)(d)} = \mathbf{R}'(v'_{(a)}, v'_{(b)}, v'_{(c)}, v'_{(d)}), \\ (d\mathbf{R})_{(a)(b)(c)(d)} &= \mathbf{R}'_{(a)(b)(c)(d)} - \mathbf{R}_{(a)(b)(c)(d)}. \end{aligned}$$

Clearly, $\Gamma_{(a)(b)(4)} = \Gamma'_{(a)(b)(4)} = 0$. We use now the definition of the Riemann curvature tensor to find a system of equations for $\underline{L}[(d\Gamma)_{(a)(b)(c)}]$. We have

$$\begin{aligned} \mathbf{R}_{(a)(b)(c)(d)} &= \mathbf{g}(v_{(a)}, \mathbf{D}_{v_{(c)}}(\mathbf{D}_{v_{(d)}}v_{(b)}) - \mathbf{D}_{v_{(d)}}(\mathbf{D}_{v_{(c)}}v_{(b)}) - \mathbf{D}_{[v_{(c)}, v_{(d)}]}v_{(b)}) \\ &= \mathbf{g}(v_{(a)}, \mathbf{D}_{v_{(c)}}(\mathbf{g}^{(m)(n)}\Gamma_{(m)(b)(d)}v_{(n)})) - \mathbf{g}(v_{(a)}, \mathbf{D}_{v_{(d)}}(\mathbf{g}^{(m)(n)}\Gamma_{(m)(b)(c)}v_{(n)})) \\ &\quad + \mathbf{g}^{(m)(n)}\Gamma_{(a)(b)(n)}(\Gamma_{(m)(c)(d)} - \Gamma_{(m)(d)(c)}) \\ &= v_{(c)}(\Gamma_{(a)(b)(d)}) - v_{(d)}(\Gamma_{(a)(b)(c)}) + \mathbf{g}^{(m)(n)}\Gamma_{(a)(b)(n)}(\Gamma_{(m)(c)(d)} - \Gamma_{(m)(d)(c)}) \\ &\quad + \mathbf{g}_{(a)(n)}[\Gamma_{(m)(b)(d)}v_{(c)}(\mathbf{g}^{(m)(n)}) - \Gamma_{(m)(b)(c)}v_{(d)}(\mathbf{g}^{(m)(n)})] \\ &\quad + \mathbf{g}^{(m)(n)}(\Gamma_{(m)(b)(d)}\Gamma_{(a)(n)(c)} - \Gamma_{(m)(b)(c)}\Gamma_{(a)(n)(d)}). \end{aligned}$$

We set $d = 4$ and use $\Gamma_{(a)(b)(4)} = v_{(4)}(\mathbf{g}^{(a)(b)}) = 0$ and $\mathbf{g}^{(a)(b)} = h^{(a)(b)}$; the result is

$$\underline{L}(\Gamma_{(a)(b)(c)}) = -h^{(m)(n)}\Gamma_{(a)(b)(n)}\Gamma_{(m)(4)(c)} - \mathbf{R}_{(a)(b)(c)(4)}.$$

Similarly,

$$\underline{L}(\Gamma'_{(a)(b)(c)}) = -h^{(m)(n)}\Gamma'_{(a)(b)(n)}\Gamma'_{(m)(4)(c)} - \mathbf{R}'_{(a)(b)(c)(4)}.$$

We subtract these two identities to derive

$$\underline{L}[(d\Gamma)_{(a)(b)(c)}] = {}^{(1)}F_{(a)(b)(c)}^{(d)(e)(f)}(d\Gamma)_{(d)(e)(f)} - (dR)_{(a)(b)(c)(4)} \quad (4.5)$$

for some smooth function ${}^{(1)}F$. This can be written schematically in the form

$$\underline{L}(d\Gamma) = \mathcal{M}_\infty(d\Gamma) + \mathcal{M}_\infty(dR). \quad (4.6)$$

We will use such schematic equations for simplicity of notation⁴.

For $a, b, c = 1, \dots, 4$ and $\alpha = 0, \dots, 3$ we define

$$\begin{aligned} (\partial d\Gamma)_{\alpha(a)(b)(c)} &= \partial_\alpha[(d\Gamma)_{(a)(b)(c)}]; \\ (\partial dR)_{\alpha(a)(b)(c)(d)} &= \partial_\alpha[(dR)_{(a)(b)(c)(d)}], \end{aligned}$$

where ∂_α are the coordinate vector-fields relative to our local coordinates in $\mathcal{O}(x_0)$. By differentiating (4.6),

$$\underline{L}(\partial d\Gamma) = \mathcal{M}_\infty(d\Gamma) + \mathcal{M}_\infty(\partial d\Gamma) + \mathcal{M}_\infty(dR) + \mathcal{M}_\infty(\partial dR). \quad (4.7)$$

Assume now that

$$\begin{aligned} v_{(a)} &= v_{(a)}^\alpha \partial_\alpha, & v'_{(a)} &= v'_{(a)}^\alpha \partial_\alpha, \\ v'_{(a)} - v_{(a)} &= (dv)_{(a)}^\alpha \partial_\alpha, & (dv)_{(a)}^\alpha &= v'_{(a)}^\alpha - v_{(a)}^\alpha, \end{aligned}$$

are the representations of the vectors $v_{(a)}$, $v'_{(a)}$, and $v'_{(a)} - v_{(a)}$ in our coordinate frame $\{\partial_\alpha\}_{\alpha=0,\dots,3}$. Since $[v_{(4)}, v_{(b)}] = -\mathbf{D}_{v_{(b)}}v_{(4)} = -\Gamma_{(4)(b)}^{(c)}v_{(c)}$, we have

$$v_{(4)}^\alpha \partial_\alpha(v_{(b)}^\beta) - v_{(b)}^\alpha \partial_\alpha(v_{(4)}^\beta) = -\Gamma_{(a)(4)(b)}v_{(c)}^\beta \mathbf{g}^{(a)(c)}.$$

⁴In general, given $B = (B_1, \dots, B_L) : \mathcal{O}(x_0) \rightarrow \mathbb{R}^L$ we let $\mathcal{M}_\infty(B) : \mathcal{O}(x_0) \rightarrow \mathbb{R}^{L'}$ denote vector-valued functions of the form $\mathcal{M}_\infty(B)_{i'} = \sum_{l=1}^L A_{li'}^l B_l$, where the coefficients $A_{li'}^l$ are smooth on $\mathcal{O}(x_0)$.

Similarly,

$$v_{(4)}^\alpha \partial_\alpha (v_{(b)}'^\beta) - v_{(b)}'^\alpha \partial_\alpha (v_{(4)}^\beta) = -\Gamma'_{(a)(4)(b)} v_{(c)}'^\beta \mathbf{g}'^{(a)(c)}.$$

We subtract these two identities to conclude that, schematically,

$$\underline{L}(dv) = \mathcal{M}_\infty(d\Gamma) + \mathcal{M}_\infty(dv). \quad (4.8)$$

As before, we define

$$(\partial dv)_{\alpha(b)}^\beta = \partial_\alpha [(dv)_{(b)}^\beta].$$

By differentiating (4.8) we have

$$\underline{L}(\partial dv) = \mathcal{M}_\infty(d\Gamma) + \mathcal{M}_\infty(\partial d\Gamma) + \mathcal{M}_\infty(dv) + \mathcal{M}_\infty(\partial dv). \quad (4.9)$$

Finally, we derive a wave equation for dR . We start from the identity

$$(\square \mathbf{g} \mathbf{R})_{(a)(b)(c)(d)} - (\square \mathbf{g}' \mathbf{R}')_{(a)(b)(c)(d)} = \mathcal{M}_\infty(dR),$$

which follows from the standard wave equations satisfied by \mathbf{R} and \mathbf{R}' and the fact that $\mathbf{g}^{(m)(n)} = \mathbf{g}'^{(m)(n)} = h^{(m)(n)}$. We also have

$$\begin{aligned} & \mathbf{D}_{(m)} \mathbf{R}_{(a)(b)(c)(d)} - \mathbf{D}'_{(m)} \mathbf{R}'_{(a)(b)(c)(d)} \\ &= \mathcal{M}_\infty(dv) + \mathcal{M}_\infty(d\Gamma) + \mathcal{M}_\infty(d\mathbf{R}) + \mathcal{M}_\infty(\partial d\mathbf{R}). \end{aligned}$$

It follows from the last two equations that

$$\begin{aligned} & \mathbf{g}^{(m)(n)} v_{(n)} (v_{(m)} (\mathbf{R}_{(a)(b)(c)(d)})) - \mathbf{g}'^{(m)(n)} v'_{(n)} (v'_{(m)} (\mathbf{R}'_{(a)(b)(c)(d)})) \\ &= \mathcal{M}_\infty(dv) + \mathcal{M}_\infty(d\Gamma) + \mathcal{M}_\infty(\partial d\Gamma) + \mathcal{M}_\infty(dR) + \mathcal{M}_\infty(\partial dR). \end{aligned}$$

Since $\mathbf{g}^{(m)(n)} = \mathbf{g}'^{(m)(n)}$ it follows that

$$\begin{aligned} & \mathbf{g}^{(m)(n)} v_{(n)} (v_{(m)} ((dR)_{(a)(b)(c)(d)})) \\ &= \mathcal{M}_\infty(dv) + \mathcal{M}_\infty(\partial dv) + \mathcal{M}_\infty(d\Gamma) + \mathcal{M}_\infty(\partial d\Gamma) + \mathcal{M}_\infty(dR) + \mathcal{M}_\infty(\partial dR). \end{aligned}$$

Thus

$$\square_{\mathbf{g}}(dR) = \mathcal{M}_\infty(dv) + \mathcal{M}_\infty(\partial dv) + \mathcal{M}_\infty(d\Gamma) + \mathcal{M}_\infty(\partial d\Gamma) + \mathcal{M}_\infty(dR) + \mathcal{M}_\infty(\partial dR). \quad (4.10)$$

This is our main wave equation.

We collect now equations (4.6), (4.7), (4.8), (4.9), and (4.10):

$$\begin{aligned} \underline{L}(d\Gamma) &= \mathcal{M}_\infty(d\Gamma) + \mathcal{M}_\infty(dR); \\ \underline{L}(\partial d\Gamma) &= \mathcal{M}_\infty(d\Gamma) + \mathcal{M}_\infty(\partial d\Gamma) + \mathcal{M}_\infty(dR) + \mathcal{M}_\infty(\partial dR); \\ \underline{L}(dv) &= \mathcal{M}_\infty(dv) + \mathcal{M}_\infty(d\Gamma); \\ \underline{L}(\partial dv) &= \mathcal{M}_\infty(dv) + \mathcal{M}_\infty(\partial dv) + \mathcal{M}_\infty(d\Gamma) + \mathcal{M}_\infty(\partial d\Gamma); \\ \square_{\mathbf{g}}(dR) &= \mathcal{M}_\infty(dv) + \mathcal{M}_\infty(\partial dv) + \mathcal{M}_\infty(d\Gamma) + \mathcal{M}_\infty(\partial d\Gamma) + \mathcal{M}_\infty(dR) + \mathcal{M}_\infty(\partial dR). \end{aligned} \quad (4.11)$$

This is our main system of equations. Since $\mathbf{g} = \mathbf{g}'$ in $I^{++} \cup I^{--}$, it follows easily that the functions $d\Gamma$, $\partial d\Gamma$, dv , ∂dv and dR vanish also in $I^{++} \cup I^{--}$. Therefore, the proposition follows from Lemma 4.4 below. \square

Lemma 4.4. *Assume $G_i, H_j : \mathcal{O}(x_0) \rightarrow \mathbb{R}$ are smooth functions, $i = 1, \dots, I$, $j = 1, \dots, J$. Let $G = (G_1, \dots, G_I)$, $H = (H_1, \dots, H_J)$, $\partial G = (\partial_0 G_1, \dots, \partial_4 G_I)$ and assume that in $\mathcal{O}(x_0)$,*

$$\begin{cases} \square_{\mathbf{g}} G = \mathcal{M}_\infty(G) + \mathcal{M}_\infty(\partial G) + \mathcal{M}_\infty(H); \\ \underline{L}(H) = \mathcal{M}_\infty(G) + \mathcal{M}_\infty(\partial G) + \mathcal{M}_\infty(H). \end{cases} \quad (4.12)$$

Assume that $G = 0$ and $H = 0$ on $(\mathcal{N} \cup \underline{\mathcal{N}}) \cap \mathcal{O}(x_0)$. Then, there exists a small neighborhood $\mathcal{O}'(x_0) \subset \mathcal{O}(x_0)$ of x_0 such that $G = 0$ and $H = 0$ in $(I^{+-} \cup I^{-+}) \cap \mathcal{O}'(x_0)$.

Unique continuation theorems of this type in the case $H = 0$ were proved by two of the authors in [10] and [11], using Carleman estimates. It is not hard to adapt the proofs, using similar Carleman estimates, to the general case; we provide all the details in the appendix. This completes the proof of Theorem 1.1.

We show now that the Killing vector-field \mathbf{K} is timelike, in a quantitative sense, in a small neighborhood of S in the complement of $I^{++} \cup I^{--}$.

Proposition 4.5. *Let \mathbf{K} be the Killing vector-field, constructed above, in a neighborhood \mathcal{O} of S . Then there is a neighborhood $\mathcal{O}' \subset \mathcal{O}$ of S such that*

$$\mathbf{g}(\mathbf{K}, \mathbf{K}) \leq \underline{u}\underline{u} \quad \text{in } (I^{+-} \cup I^{-+}) \cap \mathcal{O}'. \quad (4.13)$$

In particular, the vector-field \mathbf{K} is timelike in the set $\mathcal{O}' \setminus (I^{++} \cup I^{--})$.

Proof of Proposition 4.5. Since \mathbf{K} is a Killing vector-field in \mathcal{O} , we have

$$\square_{\mathbf{g}}(\mathbf{K}^\beta \mathbf{K}_\beta) = 2\mathbf{D}^\alpha(\mathbf{K}^\beta \mathbf{D}_\alpha \mathbf{K}_\beta) = 2\mathbf{D}^\alpha \mathbf{K}^\beta \mathbf{D}_\alpha \mathbf{K}_\beta = -4 \quad \text{on } S. \quad (4.14)$$

Indeed, $\square_{\mathbf{g}} \mathbf{K} = 0$ and it follows from (3.4) that $2\mathbf{D}^\alpha \mathbf{K}^\beta \mathbf{D}_\alpha \mathbf{K}_\beta = 4\mathbf{D}^3 \mathbf{K}^4 \mathbf{D}_3 \mathbf{K}_4 = -4$ on S . Since $\mathbf{K}_\beta \mathbf{K}^\beta = 0$ on $(\mathcal{N} \cup \underline{\mathcal{N}}) \cap \mathcal{O}$ (see (3.1)), we have $\mathbf{K}_\beta \mathbf{K}^\beta = \underline{u}\underline{u}f$ on \mathcal{O} for some smooth function $f : \mathcal{O} \rightarrow \mathbb{R}$. Using (4.14) on S and the fact that $u = \underline{u} = 0$ on S , we derive

$$-4 = \mathbf{D}^\alpha \mathbf{D}_\alpha(\underline{u}\underline{u}f) = 2f \mathbf{D}^\alpha \underline{u} \mathbf{D}_\alpha \underline{u} = -2f \underline{L}(u) \underline{L}(\underline{u}) = -2f.$$

Thus $f = 2$ on S , and the bound (4.13) follows for a sufficiently small \mathcal{O}' . \square

5. FURTHER RESULTS IN THE PRESENCE OF A SYMMETRY

The goal of this section is to prove Theorem 1.2. So far we have constructed a smooth Killing vector-field \mathbf{K} defined in an open set \mathcal{O} such that $\mathbf{K} = \underline{u}L - u\underline{L}$ on $(\mathcal{N} \cup \underline{\mathcal{N}}) \cap \mathcal{O}$.

Assume in this section that the space-time $(\mathcal{O}, \mathbf{g})$ admits another smooth Killing vector-field \mathbf{T} , which is tangent to the null hypersurfaces \mathcal{N} and $\underline{\mathcal{N}}$. We recall several definitions (see [10, Section 4] for a longer discussion and proofs of some identities). In \mathcal{O} we define the 2-form $F_{\alpha\beta} = \mathbf{D}_\alpha \mathbf{T}_\beta$ and the complex valued 2-form,

$$\mathcal{F}_{\alpha\beta} = F_{\alpha\beta} + i {}^* F_{\alpha\beta} = F_{\alpha\beta} + (i/2) \epsilon_{\alpha\beta}{}^{\mu\nu} F_{\mu\nu}. \quad (5.1)$$

Let $\mathcal{F}^2 = \mathcal{F}_{\alpha\beta} \mathcal{F}^{\alpha\beta}$. We define also the Ernst 1-form

$$\sigma_\mu = 2\mathbf{T}^\alpha \mathcal{F}_{\alpha\mu} = \mathbf{D}_\mu(-\mathbf{T}^\alpha \mathbf{T}_\alpha) - i \epsilon_{\mu\beta\gamma\delta} \mathbf{T}^\beta \mathbf{D}^\gamma \mathbf{T}^\delta. \quad (5.2)$$

It is easy to check that, in \emptyset

$$\begin{cases} \mathbf{D}_\mu \sigma_\nu - \mathbf{D}_\nu \sigma_\mu = 0; \\ \mathbf{D}^\mu \sigma_\mu = -\mathcal{F}^2; \\ \sigma_\mu \sigma^\mu = \mathbf{g}(\mathbf{T}, \mathbf{T}) \mathcal{F}^2. \end{cases} \quad (5.3)$$

Proposition 5.1. *There is an open set $\emptyset' \subseteq \emptyset$, $S \subseteq \emptyset'$ such that*

$$[\mathbf{T}, \mathbf{K}] = 0 \text{ in } \emptyset'. \quad (5.4)$$

In addition, if $\sigma_\mu = 2\mathbf{T}^\alpha \mathcal{F}_{\alpha\mu}$ is the Ernst 1-form associated to \mathbf{T} (see (5.2)), then

$$\mathbf{K}^\mu \sigma_\mu = 0 \text{ in } \emptyset'. \quad (5.5)$$

Proof of Proposition 5.1. We show first that

$$[\mathbf{T}, \mathbf{K}] = 0 \quad \text{on } (\mathcal{N} \cup \underline{\mathcal{N}}) \cap \emptyset. \quad (5.6)$$

By symmetry, it suffices to check that $[\mathbf{T}, \mathbf{K}] = 0$ on $\mathcal{N} \cap \emptyset$. We first observe that $[\mathbf{T}, L]$ is proportional to L . Indeed, since the null second fundamental form of \mathcal{N} is symmetric and \mathbf{T} is both Killing and tangent to \mathcal{N} , we have for every $X \in T(\mathcal{N})$,

$$\begin{aligned} \mathbf{g}([\mathbf{T}, L], X) &= \mathbf{g}(\mathbf{D}_\mathbf{T} L, X) - \mathbf{g}(\mathbf{D}_L \mathbf{T}, X) = \mathbf{g}(\mathbf{D}_\mathbf{T} L, X) + \mathbf{g}(\mathbf{D}_X \mathbf{T}, L) \\ &= \mathbf{g}(\mathbf{D}_\mathbf{T} L, X) - \mathbf{g}(\mathbf{T}, \mathbf{D}_X L) = \chi(\mathbf{T}, X) - \chi(X, \mathbf{T}) = 0. \end{aligned}$$

Consequently $[\mathbf{T}, L]$ must be proportional to L , i.e. $[\mathbf{T}, L] = fL$. Since $\mathbf{D}_L L = 0$ and \mathbf{T} commutes with covariant derivatives we derive,

$$\begin{aligned} 0 &= \mathcal{L}_\mathbf{T}(\mathbf{D}_L L) = \mathbf{D}_{\mathcal{L}_\mathbf{T} L} L + \mathbf{D}_L(\mathcal{L}_\mathbf{T} L) \\ &= \mathbf{D}_{fL} L + \mathbf{D}_L(fL) = L(f)L. \end{aligned}$$

Therefore

$$[\mathbf{T}, L] = fL \quad \text{and} \quad L(f) = 0 \quad \text{on } \mathcal{N} \cap \emptyset. \quad (5.7)$$

On the other hand, in view of the definition of \underline{u} we have $\mathbf{T}(L(\underline{u})) - L(\mathbf{T}(\underline{u})) = fL(\underline{u})$. Hence,

$$L(f\underline{u} + \mathbf{T}(\underline{u})) = 0.$$

Since \mathbf{T} is tangent to S and $\underline{u} = 0$ on S , we deduce that $f\underline{u} + \mathbf{T}(\underline{u})$ vanishes on S , thus

$$\mathbf{T}\underline{u} + f\underline{u} = 0, \quad \text{on } \mathcal{N} \cap \emptyset.$$

Now, $[\mathbf{T}, \underline{u}L] = \mathbf{T}(\underline{u})L + \underline{u}[\mathbf{T}, L] = (\mathbf{T}(\underline{u}) + f\underline{u})L = 0$. The identity (5.6) follows since $\mathbf{K} = \underline{u}L$ on $\mathcal{N} \cap \emptyset$.

Let $V = [\mathbf{T}, \mathbf{K}] = \mathcal{L}_\mathbf{T} \mathbf{K}$ on \emptyset . Since $\square_\mathbf{g} \mathbf{K} = 0$ and \mathbf{T} is Killing, we derive, after commuting covariant and Lie derivatives,

$$0 = \mathcal{L}_\mathbf{T}(\square_\mathbf{g} \mathbf{K}) = \square_\mathbf{g}(\mathcal{L}_\mathbf{T} \mathbf{K}) = \square_\mathbf{g} V.$$

Since V vanishes on $(\mathcal{N} \cup \underline{\mathcal{N}}) \cap \emptyset$, it follows that V vanishes in $(I^{++} \cup I^{--}) \cap \emptyset'$, for some smaller neighborhood \emptyset' of S . (due to the well-posedness of the characteristic initial-value

problem); it also follows that V vanishes in $(I^{+-} \cup I^{-+}) \cap \mathcal{O}'$ using Lemma 4.4 with $H = 0$. This completes the proof of (5.4).

We prove now the identity (5.5). Since \mathbf{K} and \mathbf{T} commute we observe that $\mathcal{L}_{\mathbf{K}}\mathcal{F} = 0$ in \mathcal{O} . In addition, since $\square_{\mathbf{g}}\mathbf{K} = 0$, \mathbf{DK} is antisymmetric, $\mathbf{D}\sigma$ is symmetric with trace $\mathbf{D}^\alpha\sigma_\alpha = -\mathcal{F}^2$ (see (5.3)) and $\mathbf{Ric}(\mathbf{g}) = 0$, we have in \mathcal{O}

$$\square_{\mathbf{g}}(\mathbf{K}^\mu\sigma_\mu) = \mathbf{K}^\mu\square_{\mathbf{g}}\sigma_\mu = \mathbf{K}^\mu\mathbf{D}_\mu(\mathbf{D}^\alpha\sigma_\alpha) = -\mathcal{L}_{\mathbf{K}}\mathcal{F}^2 = 0. \quad (5.8)$$

We show below that the function $\mathbf{K}^\mu\sigma_\mu$ vanishes on $(\mathcal{N} \cup \underline{\mathcal{N}}) \cap \mathcal{O}$. Thus, as before, we conclude that $\mathbf{K}^\mu\sigma_\mu = 0$ in a smaller neighborhood \mathcal{O}' , as desired.

To show that $\mathbf{K}^\mu\sigma_\mu$ vanishes on $(\mathcal{N} \cup \underline{\mathcal{N}}) \cap \mathcal{O}$ we calculate with respect to our null frame $L = e_4$, $\underline{L} = e_3$, e_1 , e_2 defined in a neighborhood of S . Since \mathbf{T} is tangent to \mathcal{N} , for $a = 1, 2$ we have $F_{a4} = e_a(\mathbf{g}(\mathbf{T}, e_4)) - \mathbf{g}(\mathbf{T}, \mathbf{D}_{e_a}e_4) = 0$ along \mathcal{N} (since $\mathbf{D}_{e_a}e_4 = -\zeta_a e_4$, see (2.8)). Similarly, $F_{a3} = 0$ along $\underline{\mathcal{N}}$. Thus

$$\mathcal{F}_{14} = \mathcal{F}_{24} = 0 \text{ on } \mathcal{N} \cap \mathcal{O} \quad \text{and} \quad \mathcal{F}_{13} = \mathcal{F}_{23} = 0 \text{ on } \underline{\mathcal{N}} \cap \mathcal{O}. \quad (5.9)$$

Since $\mathbf{K} = \underline{u}e_4 - ue_3$ on $(\mathcal{N} \cup \underline{\mathcal{N}}) \cap \mathcal{O}$, we infer that,

$$\mathbf{K}^\mu\sigma_\mu = 2\mathbf{K}^\mu\mathbf{T}^\alpha\mathcal{F}_{\alpha\mu} = 0 \quad \text{on } (\mathcal{N} \cup \underline{\mathcal{N}}) \cap \mathcal{O}, \quad (5.10)$$

as desired. \square

Proposition 5.2. *There is a constant $\lambda_0 \in \mathbb{R}$ and an open neighborhood $\mathcal{O}' \subseteq \mathcal{O}$ of S such that the vector-field*

$$\mathbf{Z} = \mathbf{T} + \lambda_0\mathbf{K}$$

has periodic orbits in \mathcal{O}' . In other words, there is $t_0 > 0$ such that $\Psi_{t_0, \mathbf{Z}} = \text{Id}$ in \mathcal{O}' .

This completes the proof of Theorem 1.2. Observe that the main constants λ_0 and t_0 can be determined on the bifurcation sphere S . We show below that Proposition 5.2 follows from the following lemma.

Lemma 5.3. *There is a constant $t_0 > 0$ such that $\Psi_{t_0, \mathbf{T}} = \text{Id}$ in S . In addition, there is a constant $\lambda_0 \in \mathbb{R}$ and a choice of the null pair (L, \underline{L}) along S (satisfying (2.1)) such that*

$$[\mathbf{T}, L] = \lambda_0 L \quad \text{and} \quad [\mathbf{T}, \underline{L}] = -\lambda_0 \underline{L} \quad \text{on } S. \quad (5.11)$$

Proof of Proposition 5.2. It follows from (5.7) and (5.11) that

$$[\mathbf{T}, L] = \lambda_0 L \quad \text{on } \mathcal{N} \cap \mathcal{O} \quad \text{and} \quad [\mathbf{T}, \underline{L}] = -\lambda_0 \underline{L} \quad \text{on } \underline{\mathcal{N}} \cap \mathcal{O}. \quad (5.12)$$

Thus, using the identity $[\underline{L}, \mathbf{K}] = -\underline{L}$ in Proposition 3.1,

$$[\mathbf{Z}, \underline{L}] = [\mathbf{T} + \lambda_0\mathbf{K}, \underline{L}] = 0 \quad \text{on } \underline{\mathcal{N}} \cap \mathcal{O}.$$

Since \mathbf{Z} is a Killing vector-field, it follows as in the proof of Proposition 3.1 (see (3.15)) that

$$[\mathbf{Z}, \underline{L}] = 0 \quad \text{in } \mathcal{O}.$$

An argument similar to the proof of (3.16) shows that $[L, \mathbf{K}] - L = 0$ on $\underline{\mathcal{N}} \cap \mathcal{O}$. Using the first identity in (5.12), it follows that $[\mathbf{Z}, L] = 0$ on $\underline{\mathcal{N}} \cap \mathcal{O}$. Since \mathbf{Z} is a Killing vector-field, it follows as in Proposition 3.1 that $[\mathbf{Z}, L] = 0$ in \mathcal{O} .

The conclusion of the proposition follows from the first claim in Lemma 5.3 and the identities $[\mathbf{Z}, \underline{L}] = [\mathbf{Z}, L] = 0$ in \mathcal{O} . \square

Proof of Lemma 5.3. The existence of the period t_0 is a standard fact concerning Killing vector-fields on the sphere⁵. In particular all nontrivial orbits of S are compact and diffeomorphic to \mathbb{S}^1 . To prove (5.11), in view of (5.7) it suffices to prove that there is $\lambda_0 \in \mathbb{R}$ and a choice of the null pair (L, \underline{L}) on S such that

$$\mathbf{g}([\mathbf{T}, L], \underline{L}) = -\lambda_0, \quad \mathbf{g}([\mathbf{T}, \underline{L}], L) = \lambda_0 \quad \text{on } S.$$

Both identities are equivalent to

$$\mathbf{T}^\alpha \underline{L}^\beta \mathbf{D}_\alpha L_\beta - L^\alpha \underline{L}^\beta \mathbf{D}_\alpha \mathbf{T}_\beta = -\lambda_0,$$

which is equivalent to

$$\lambda_0 = F_{43} - \mathbf{g}(\zeta, \mathbf{T}).$$

We thus have to show that there exist a choice of the null pair $e_4 = L, e_3 = \underline{L}$ along S such that the scalar function below is constant along S ,

$$H := F_{43} - \mathbf{g}(\zeta, \mathbf{T}). \quad (5.13)$$

Under a scaling transformation $e'_4 = f e_4, e'_3 = f^{-1} e_3$ the torsion ζ changes according to the formula,

$$\zeta' = \zeta - \nabla \log f.$$

Therefore, in the new frame,

$$H' = F_{4'3'} - \mathbf{g}(\zeta', \mathbf{T}) = F_{43} - \mathbf{g}(\zeta, \mathbf{T}) + \mathbf{T}(\log f) = H + \mathbf{T}(\log f)$$

Consequently, we are led to look for a function f such that $H + \mathbf{T}(\log f)$ is a constant. Taking \hat{H} to be the average of H along the integral curves of \mathbf{T} and solving the equation

$$\mathbf{T}(\log f) = -H + \hat{H}, \quad (5.14)$$

it only remains to prove that \hat{H} is constant along S .

Since \mathbf{T} is Killing we must have,

$$\mathbf{D}_\alpha \mathbf{D}_\beta T_\gamma = T^\lambda \mathbf{R}_{\lambda\alpha\beta\gamma} \quad (5.15)$$

Using (5.15) and the formulas (2.8) on S we derive,

$$\mathbf{T}^\lambda \mathbf{R}_{\lambda a 43} = \mathbf{D}_a \mathbf{D}_4 \mathbf{T}_3 = e_a(\mathbf{D}_4 \mathbf{T}_3) = e_a(F_{43}).$$

Thus, since \mathbf{T} is tangent to S and $\mathbf{T}^b \mathbf{R}_{ba 43} = \frac{1}{2} \in_{ab} \mathbf{T}^b \sigma$ (with $\sigma = {}^* \mathbf{R}_{3434}$)

$$e_a(F_{43}) = \mathbf{T}^b \mathbf{R}_{ba 43} = \frac{1}{2} \in_{ab} \mathbf{T}^b \sigma. \quad (5.16)$$

In particular, the function H defined in (5.13) is constant on S if $\mathbf{T} \equiv 0$ on S . Thus we may assume in the rest of the proof that the set $\Lambda = \{p \in S : \mathbf{T}_p = 0\}$ is finite.

⁵If $\mathbf{T} \equiv 0$ on S then any value of $t_0 > 0$ is suitable. In this case, the conclusion of Proposition 5.2 is that $\mathbf{T} + \lambda_0 \mathbf{K} \equiv 0$ in \mathcal{O}' for some $\lambda_0 \in \mathbb{R}$.

On the other hand, writing $\nabla_a \zeta_b - \nabla_b \zeta_a = \epsilon_{ab} \text{curl} \zeta$,

$$\begin{aligned} e_a \mathbf{g}(\zeta, \mathbf{T}) &= \nabla_a \zeta_b \mathbf{T}^b + \zeta_b \nabla_a \mathbf{T}_b = (\nabla_a \zeta_b - \nabla_b \zeta_a) \mathbf{T}^b + \zeta^b \nabla_a \mathbf{T}_b + \nabla_{\mathbf{T}} \zeta_a \\ &= \epsilon_{ab} \text{curl} \zeta \mathbf{T}^b + \zeta^b \nabla_a \mathbf{T}_b + \nabla_{\mathbf{T}} \zeta_a \end{aligned}$$

The torsion ζ verifies the equation,

$$\text{curl} \zeta = \frac{1}{2} \sigma, \quad (5.17)$$

Therefore,

$$e_a \mathbf{g}(\zeta, \mathbf{T}) = \frac{1}{2} \epsilon_{ab} \mathbf{T}^b \sigma + \zeta^b \nabla_a \mathbf{T}_b + \nabla_{\mathbf{T}} \zeta_a. \quad (5.18)$$

Since $H = F_{43} - \zeta \mathbf{T}$ we deduce,

$$e_a(H) = -\zeta^b \nabla_a \mathbf{T}_b - \nabla_{\mathbf{T}} \zeta_a. \quad (5.19)$$

Consider the orthonormal frame e_1, e_2 on $S \setminus \Lambda$,

$$e_1 = X^{-1} \mathbf{T}, \quad X^2 = \mathbf{g}(\mathbf{T}, \mathbf{T}).$$

Since $e_1(X) = 0$ and $e_1 = X^{-1} \mathbf{T}$, we have

$$\nabla_{\mathbf{T}} e_2 = -F_{12} e_1.$$

We claim that, with respect to this local frame,

$$\nabla_2(H) = -\mathbf{T}(\zeta_2). \quad (5.20)$$

Indeed,

$$\begin{aligned} \nabla_2(H) &= -\zeta^1 \nabla_2 \mathbf{T}_1 - \zeta^2 \nabla_2 \mathbf{T}_2 - \mathbf{g}(\nabla_{\mathbf{T}} \zeta, e_2) \\ &= -\zeta^1 F_{21} - \mathbf{T} \mathbf{g}(\zeta, e_2) + \mathbf{g}(\zeta, \nabla_{\mathbf{T}} e_2) \\ &= -\mathbf{T} \mathbf{g}(\zeta, e_2) - \zeta^1 F_{21} - \zeta^1 F_{12} \\ &= -\mathbf{T}(\zeta_2) \end{aligned}$$

We now fix a non-trivial orbit γ_0 of \mathbf{T} in $S \setminus \Lambda$. Consider the geodesics initiating on γ_0 and perpendicular to it and ϕ the corresponding affine parameter. More precisely we choose a vector V on γ_0 such that $\mathbf{g}(V, V) = 1$ and extend it by parallel transport along the geodesics perpendicular to γ_0 . Then choose ϕ such that $V(\phi) = 1$ and $\phi = 0$ on γ_0 . This defines a system of coordinates t, ϕ in a neighborhood U of γ_0 , such that $\partial_t = T$, $\nabla_{\partial_\phi} \partial_\phi = 0$ in U and $\mathbf{g}(\partial_t, \partial_\phi) = 0$, $\mathbf{g}(\partial_\phi, \partial_\phi) = 1$ on Γ_0 . Since ∂_t is Killing we must have $X^2 = -\mathbf{g}(\partial_t, \partial_t)$ and $\mathbf{g}(\partial_\phi, \partial_\phi)$ independent of t . Moreover,

$$\partial_\phi \mathbf{g}(\partial_t, \partial_\phi) = \mathbf{g}(\nabla_{\partial_\phi} \partial_t, \partial_\phi) + \mathbf{g}(\partial_t, \nabla_{\partial_\phi} \partial_\phi) = \mathbf{g}(\nabla_{\partial_t} \partial_\phi, \partial_\phi) = \frac{1}{2} \partial_t \mathbf{g}(\partial_\phi, \partial_\phi) = 0.$$

Hence, since $\mathbf{g}(\partial_t, \partial_\phi) = 0$ on Γ_0 we infer that $\mathbf{g}(\partial_t, \partial_\phi) = 0$ in U . Similarly,

$$\partial_\phi \mathbf{g}(\partial_\phi, \partial_\phi) = 2\mathbf{g}(\nabla_{\partial_\phi} \partial_\phi, \partial_\phi) = 0$$

and therefore, $\mathbf{g}(\partial_\phi, \partial_\phi) = 1$ in U . Thus, in U , the metric \mathbf{g} takes the form,

$$d\phi^2 + X^2(\phi)dt^2 \quad (5.21)$$

Therefore, with $\mathbf{T} = \partial_t$, $e_2 = \partial_\phi$, we deduce from (5.20), everywhere in U ,

$$\partial_\phi H = -\partial_t \mathbf{g}(\zeta, \partial_\phi) \quad (5.22)$$

Thus, integrating in t and in view of the fact that the orbits of ∂_t are closed, we infer that \hat{H} is constant along S , as desired. \square

APPENDIX A. PROOF OF LEMMA 4.4

We will use a Carleman estimate proved by two of the authors in [10, Section 3], which we recall below. Let $\mathcal{O}(x_0)$ a coordinate neighborhood of a point $x_0 \in S$ and coordinates x^α as in (4.2). We denote by $B_r = B_r(x_0)$, the set of points $p \in \mathcal{O}(x_0)$ whose coordinates $x = x^\alpha$ verify $|x - x_0| \leq r$, relative to the standard euclidean norm in $\mathcal{O}(x_0)$. Consider two vector-fields $V = V^\alpha \partial_\alpha$, $W = W^\alpha \partial_\alpha$ on $\mathcal{O}(x_0)$ which verify, that,

$$\sup_{x \in \mathcal{O}(x_0)} \sum_{j=0}^4 (|\partial^j V(x)| + |\partial^j W(x)|) \leq A, \quad (A.1)$$

where A is a large constant (as in (4.2)), and $|\partial^j V(x)|$ denotes the sum of the absolute values of all partial derivatives of order j of all components of V in our given coordinate system. When $j = 1$ we write simply $|\partial V(x)|$.

Definition A.1. A family of weights $h_\epsilon : B_{\epsilon^{10}} \rightarrow \mathbb{R}_+$, $\epsilon \in (0, \epsilon_1)$, $\epsilon_1 \leq A^{-1}$, will be called *V*-conditional pseudo-convex if for any $\epsilon \in (0, \epsilon_1)$

$$h_\epsilon(x_0) = \epsilon, \quad \sup_{x \in B_{\epsilon^{10}}} \sum_{j=1}^4 \epsilon^j |\partial^j h_\epsilon(x)| \leq \epsilon/\epsilon_1, \quad |V(h_\epsilon)(x_0)| \leq \epsilon^{10}, \quad (A.2)$$

$$\mathbf{D}^\alpha h_\epsilon(x_0) \mathbf{D}^\beta h_\epsilon(x_0) (\mathbf{D}_\alpha h_\epsilon \mathbf{D}_\beta h_\epsilon - \epsilon \mathbf{D}_\alpha \mathbf{D}_\beta h_\epsilon)(x_0) \geq \epsilon_1^2, \quad (A.3)$$

and there is $\mu \in [-\epsilon_1^{-1}, \epsilon_1^{-1}]$ such that for all vectors $X = X^\alpha \partial_\alpha$ at x_0

$$\begin{aligned} & \epsilon_1^2 [(X^1)^2 + (X^2)^2 + (X^3)^2 + (X^4)^2] \\ & \leq X^\alpha X^\beta (\mu \mathbf{g}_{\alpha\beta} - \mathbf{D}_\alpha \mathbf{D}_\beta h_\epsilon)(x_0) + \epsilon^{-2} (|X^\alpha V_\alpha(x_0)|^2 + |X^\alpha \mathbf{D}_\alpha h_\epsilon(x_0)|^2). \end{aligned} \quad (A.4)$$

A function $e_\epsilon : B_{\epsilon^{10}} \rightarrow \mathbb{R}$ will be called a negligible perturbation if

$$\sup_{x \in B_{\epsilon^{10}}} |\partial^j e_\epsilon(x)| \leq \epsilon^{10} \quad \text{for } j = 0, \dots, 4. \quad (A.5)$$

Our main Carleman estimate, see [10, Section 3], is the following:

Lemma A.2. Assume $\epsilon_1 \leq A^{-1}$, $\{h_\epsilon\}_{\epsilon \in (0, \epsilon_1)}$ is a *V*-conditional pseudo-convex family, and e_ϵ is a negligible perturbation for any $\epsilon \in (0, \epsilon_1]$. Then there is $\epsilon \in (0, \epsilon_1)$ sufficiently

small (depending only on ϵ_1) and \tilde{C}_ϵ sufficiently large such that for any $\lambda \geq \tilde{C}_\epsilon$ and any $\phi \in C_0^\infty(B_{\epsilon^{10}})$

$$\lambda \|e^{-\lambda f_\epsilon} \phi\|_{L^2} + \|e^{-\lambda f_\epsilon} |\partial \phi|\|_{L^2} \leq \tilde{C}_\epsilon \lambda^{-1/2} \|e^{-\lambda f_\epsilon} \square_{\mathbf{g}} \phi\|_{L^2} + \epsilon^{-6} \|e^{-\lambda f_\epsilon} V(\phi)\|_{L^2}, \quad (\text{A.6})$$

where $f_\epsilon = \ln(h_\epsilon + e_\epsilon)$.

We will only use this Carleman estimate with $V = 0$. In this case the pseudo-convexity condition in Definition A.1 is a special case of Hörmander's pseudo-convexity condition [9, Chapter 28]. We also need a Carleman estimate to exploit the ODE's in (4.12).

Lemma A.3. *Assume $\epsilon \leq A^{-1}$ is sufficiently small, e_ϵ is a negligible perturbation, and $h_\epsilon : B_{\epsilon^{10}} \rightarrow \mathbf{R}_+$ satisfies*

$$h_\epsilon(x_0) = \epsilon, \quad \sup_{x \in B_{\epsilon^{10}}} \sum_{j=1}^2 \epsilon^j |\partial^j h_\epsilon(x)| \leq 1, \quad |W(h_\epsilon)(x_0)| \geq 1. \quad (\text{A.7})$$

Then there is \tilde{C}_ϵ sufficiently large such that for any $\lambda \geq \tilde{C}_\epsilon$ and any $\phi \in C_0^\infty(B_{\epsilon^{10}})$

$$\|e^{-\lambda f_\epsilon} \phi\|_{L^2} \leq 4\lambda^{-1} \|e^{-\lambda f_\epsilon} W(\phi)\|_{L^2}, \quad (\text{A.8})$$

where $f_\epsilon = \ln(h_\epsilon + e_\epsilon)$.

Proof of Lemma A.3. Clearly, we may assume that ϕ is real-valued and let $\psi = e^{-\lambda f_\epsilon} \phi \in C_0^\infty(B_{\epsilon^{10}})$. We have to prove that

$$\|\psi\|_{L^2} \leq 4\lambda^{-1} W(\psi) + W(f_\epsilon)\psi\|_{L^2}. \quad (\text{A.9})$$

By integration by parts,

$$\begin{aligned} & \int_{B_{\epsilon^{10}}} [\lambda^{-1} W(\psi) + W(f_\epsilon)\psi] \cdot W(f_\epsilon)\psi \, d\mu \\ &= \int_{B_{\epsilon^{10}}} [W(f_\epsilon)\psi]^2 \, d\mu - (2\lambda)^{-1} \int_{B_{\epsilon^{10}}} \psi^2 \cdot \mathbf{D}_\alpha(W(f_\epsilon)W^\alpha) \, d\mu. \end{aligned}$$

In view of (A.7) and the assumption (A.1)

$$|W(f_\epsilon)| \geq 1 \quad \text{and} \quad |\mathbf{D}_\alpha(W(f_\epsilon)W^\alpha)| \leq \tilde{C}_\epsilon \quad \text{in} \quad B_{\epsilon^{10}},$$

provided that ϵ is sufficiently small. Thus, for λ sufficiently large,

$$\int_{B_{\epsilon^{10}}} [\lambda^{-1} W(\psi) + W(f_\epsilon)\psi] \cdot W(f_\epsilon)\psi \, d\mu \geq \frac{1}{2} \int_{B_{\epsilon^{10}}} [W(f_\epsilon)\psi]^2 \, d\mu,$$

and the bound (A.9) follows. \square

Proof of Lemma 4.4. It suffices to prove that $G = 0$ and $H = 0$ in $I_{\tilde{c}}^{+-}$, for some \tilde{c} sufficiently small. We fix $x_0 \in S$ and set

$$h_\epsilon = \epsilon^{-1}(u + \epsilon)(-\underline{u} + \epsilon) \quad \text{and} \quad e_\epsilon = \epsilon^{10} N^{x_0}, \quad (\text{A.10})$$

where u, \underline{u} are the optical functions defined in section 2 and $N^{x_0}(x) = |x - x_0|^2 = \sum_{\alpha=0,1,2,3} |x^\alpha - x_0^\alpha|^2$, the square of the standard euclidean norm.

It is clear that e_ϵ is a negligible perturbation, in the sense of (A.5), for ϵ sufficiently small. Also, it is clear that h_ϵ verifies the condition (A.7), for ϵ sufficiently small and $W = 2\underline{L}$.

We show now that there is $\epsilon_1 = \epsilon_1(A)$ sufficiently small such that the family of weights $\{h_\epsilon\}_{\epsilon \in (0, \epsilon_1)}$ is 0-conditional pseudo-convex, in the sense of Definition A.1. Condition (A.2) is clearly satisfied, in view of the definition and (4.3). To verify conditions (A.3) and (A.4), we compute, in the frame e_1, e_2, e_3, e_4 defined in section 2,

$$e_1(h_\epsilon) = e_2(h_\epsilon) = 0, \quad e_3(h_\epsilon) = -\Omega(1 - \epsilon^{-1}\underline{u}), \quad e_4(h_\epsilon) = \Omega(1 + \epsilon^{-1}u) \quad (\text{A.11})$$

in $B_{\epsilon^{10}}(x_0)$, and

$$\begin{aligned} (\mathbf{D}^2 h_\epsilon)_{ab} &= O(1), & (\mathbf{D}^2 h_\epsilon)_{3a} &= O(1), & (\mathbf{D}^2 h_\epsilon)_{4a} &= O(1), & a, b &= 1, 2, \\ (\mathbf{D}^2 h_\epsilon)_{33} &= O(1), & (\mathbf{D}^2 h_\epsilon)_{44} &= O(1), & (\mathbf{D}^2 h_\epsilon)_{34} &= -\Omega^2 \epsilon^{-1} + O(1) \end{aligned} \quad (\text{A.12})$$

in $B_{\epsilon^{10}}(x_0)$, where $O(1)$ denotes various functions on $B_{\epsilon^{10}}(x_0)$ with absolute value bounded by constants that depends only on A . Thus

$$\mathbf{D}^\alpha h_\epsilon(x_0) \mathbf{D}^\beta h_\epsilon(x_0) (\mathbf{D}_\alpha h_\epsilon \mathbf{D}_\beta h_\epsilon - \epsilon \mathbf{D}_\alpha \mathbf{D}_\beta h_\epsilon)(x_0) = 2 + \epsilon O(1).$$

This proves (A.3) if ϵ_1 is sufficiently small. Similarly, if $X = X^\alpha e_\alpha$ then, with $\mu = \epsilon_1^{-1/2}$ we compute

$$\begin{aligned} & X^\alpha X^\beta (\mu \mathbf{g}_{\alpha\beta} - \mathbf{D}_\alpha \mathbf{D}_\beta h_\epsilon)(x_0) + \epsilon^{-2} |X^\alpha \mathbf{D}_\alpha h_\epsilon(x_0)|^2 \\ &= \mu((X^1)^2 + (X^2)^2) + 2(\epsilon^{-1} - \mu)X^3 X^4 + \epsilon^{-2}(X^3 - X^4)^2 + O(1) \sum_{\alpha=1}^4 (X^\alpha)^2 \\ &\geq (\mu/2)((X^1)^2 + (X^2)^2) + (\epsilon^{-1}/2)((X^3)^2 + (X^4)^2), \end{aligned}$$

provided that ϵ_1 is sufficiently small. This completes the proof of (A.4).

It follows from the Carleman estimates in Lemmas A.2 and A.3 that there is $\epsilon = \epsilon(A) \in (0, c)$ (where c is the constant in Lemma 4.4) and a constant $\tilde{C} = \tilde{C}(A) \geq 1$ such that

$$\begin{aligned} & \lambda \|e^{-\lambda f_\epsilon} \phi\|_{L^2} + \|e^{-\lambda f_\epsilon} |\partial \phi|\|_{L^2} \leq \tilde{C} \lambda^{-1/2} \|e^{-\lambda f_\epsilon} \square_{\mathbf{g}} \phi\|_{L^2}; \\ & \|e^{-\lambda f_\epsilon} \phi\|_{L^2} \leq \tilde{C} \lambda^{-1} \|e^{-\lambda f_\epsilon} \underline{L}(\phi)\|_{L^2}, \end{aligned} \quad (\text{A.13})$$

for any $\phi \in C_0^\infty(B_{\epsilon^{10}}(x_0))$ and any $\lambda \geq \tilde{C}$, where $f_\epsilon = \ln(h_\epsilon + e_\epsilon)$. Let $\eta : \mathbb{R} \rightarrow [0, 1]$ denote a smooth function supported in $[1/2, \infty)$ and equal to 1 in $[3/4, \infty)$. For $\delta \in (0, 1]$, $i = 1, \dots, I$, $j = 1, \dots, J$ we define,

$$\begin{aligned} G_i^{\delta, \epsilon} &= G_i \cdot \mathbf{1}_{I_c^{+-}} \cdot \eta(-u\underline{u}/\delta) \cdot (1 - \eta(N^{x_0}/\epsilon^{20})) = G_i \cdot \tilde{\eta}_{\delta, \epsilon} \\ H_j^{\delta, \epsilon} &= H_j \cdot \mathbf{1}_{I_c^{+-}} \cdot \eta(-u\underline{u}/\delta) \cdot (1 - \eta(N^{x_0}/\epsilon^{20})) = H_j \cdot \tilde{\eta}_{\delta, \epsilon}. \end{aligned} \quad (\text{A.14})$$

Clearly, $G_i^{\delta,\epsilon}, H_j^{\delta,\epsilon} \in C_0^\infty(B_{\epsilon^{10}}(x_0) \cap \mathbf{E})$. We would like to apply the inequalities in (A.13) to the functions $G_i^{\delta,\epsilon}, H_j^{\delta,\epsilon}$, and then let $\delta \rightarrow 0$ and $\lambda \rightarrow \infty$ (in this order).

Using the definition (A.14), we have

$$\begin{aligned}\square_{\mathbf{g}} G_i^{\delta,\epsilon} &= \tilde{\eta}_{\delta,\epsilon} \cdot \square_{\mathbf{g}} G_i + 2\mathbf{D}_\alpha G_i \cdot \mathbf{D}^\alpha \tilde{\eta}_{\delta,\epsilon} + G_i \cdot \square_{\mathbf{g}} \tilde{\eta}_{\delta,\epsilon}; \\ \underline{L}(H_j^{\delta,\epsilon}) &= \tilde{\eta}_{\delta,\epsilon} \cdot \underline{L}(H_j) + H_j \cdot \underline{L}(\tilde{\eta}_{\delta,\epsilon}).\end{aligned}$$

Using the Carleman inequalities (A.13), for any $i = 1, \dots, I, j = 1, \dots, J$ we have

$$\begin{aligned}\lambda \cdot \|e^{-\lambda f_\epsilon} \cdot \tilde{\eta}_{\delta,\epsilon} G_i\|_{L^2} + \|e^{-\lambda f_\epsilon} \cdot \tilde{\eta}_{\delta,\epsilon} |\partial^1 G_i|\|_{L^2} &\leq \tilde{C} \lambda^{-1/2} \cdot \|e^{-\lambda f_\epsilon} \cdot \tilde{\eta}_{\delta,\epsilon} \square_{\mathbf{g}} G_i\|_{L^2} \\ &+ \tilde{C} \left[\|e^{-\lambda f_\epsilon} \cdot \mathbf{D}_\alpha G_i \mathbf{D}^\alpha \tilde{\eta}_{\delta,\epsilon}\|_{L^2} + \|e^{-\lambda f_\epsilon} \cdot G_i (|\square_{\mathbf{g}} \tilde{\eta}_{\delta,\epsilon}| + |\partial^1 \tilde{\eta}_{\delta,\epsilon}|)\|_{L^2} \right]\end{aligned}\quad (\text{A.15})$$

and

$$\|e^{-\lambda f_\epsilon} \cdot \tilde{\eta}_{\delta,\epsilon} H_j\|_{L^2} \leq \tilde{C} \lambda^{-1} \|e^{-\lambda f_\epsilon} \cdot \tilde{\eta}_{\delta,\epsilon} \underline{L}(H_j)\|_{L^2} + \tilde{C} \lambda^{-1} \|e^{-\lambda f_\epsilon} \cdot H_j \underline{L}(\tilde{\eta}_{\delta,\epsilon})\|_{L^2}, \quad (\text{A.16})$$

for any $\lambda \geq \tilde{C}$. Using the main identities (4.12), in $B_{\epsilon^{10}}(x_0)$ we estimate pointwise

$$\begin{aligned}|\square_{\mathbf{g}} G_i| &\leq M \sum_{l=1}^I (|\partial^1 G_l| + |G_l|) + M \sum_{m=1}^J |H_j|, \\ |\underline{L}(H_j)| &\leq M \sum_{l=1}^I (|\partial^1 G_l| + |G_l|) + M \sum_{m=1}^J |H_j|,\end{aligned}\quad (\text{A.17})$$

for some large constant M . We add inequalities (A.15) and (A.16) over i, j . The key observation is that, in view of (A.17), the first terms in the right-hand sides of (A.15) and (A.16) can be absorbed into the left-hand sides for λ sufficiently large. Thus, for any λ sufficiently large and $\delta \in (0, 1]$,

$$\begin{aligned}\lambda \sum_{i=1}^I \|e^{-\lambda f_\epsilon} \cdot \tilde{\eta}_{\delta,\epsilon} G_i\|_{L^2} + \sum_{j=1}^J \|e^{-\lambda f_\epsilon} \cdot \tilde{\eta}_{\delta,\epsilon} H_j\|_{L^2} &\leq \tilde{C} \lambda^{-1} \sum_{j=1}^J \|e^{-\lambda f_\epsilon} \cdot H_j |\partial \tilde{\eta}_{\delta,\epsilon}|\|_{L^2} \\ &+ \tilde{C} \sum_{i=1}^I \left[\|e^{-\lambda f_\epsilon} \cdot \mathbf{D}_\alpha G_i \mathbf{D}^\alpha \tilde{\eta}_{\delta,\epsilon}\|_{L^2} + \|e^{-\lambda f_\epsilon} \cdot G_i (|\square_{\mathbf{g}} \tilde{\eta}_{\delta,\epsilon}| + |\partial \tilde{\eta}_{\delta,\epsilon}|)\|_{L^2} \right].\end{aligned}\quad (\text{A.18})$$

We let now $\delta \rightarrow 0$ and $\lambda \rightarrow \infty$, as in [10, Section 6], to conclude that $\mathbf{1}_{B_{\epsilon^{40}}(x_0) \cap I^{+-}} G_i = 0$ and $\mathbf{1}_{B_{\epsilon^{40}}(x_0) \cap I^{+-}} H_j = 0$. The main ingredient needed for this limiting procedure is the inequality

$$\inf_{B_{\epsilon^{40}}(x_0) \cap I_c^{+-}} e^{-\lambda f_\epsilon} \geq e^{\lambda/\tilde{C}} \sup_{\{x \in B_{\epsilon^{10}}(x_0) \cap I_c^{+-} : N^{x_0} \geq \epsilon^{20}/2\}} e^{-\lambda f_\epsilon},$$

which follows easily from the definition (A.10). The lemma follows. \square

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